Time-Reversal of Ultrasonic Fields—Part III: Theory of the Closed Time-Reversal Cavity

Didier Cassereau and Mathias Fink

Abstract— **A theoretical model for time-reversal cavities to optimize focusing in homogeneous and inhomogeneous media is described. The concept of the cavity can be understood as the most realistic approximation to an exact three-dimensional (3-D) time-reversal of ultrasonic fields; it is also a generalization of the time-reversal mirrors realized experimentally in our laboratory. The proposed method is based on an approach in the transient regime, that is more general than the monochromatic formalism used in optics to analyze the phase conjugation mirrors efficiency. This method uses impulse diffraction theory to obtain the impulse response of the cavity in any inhomogeneous medium. It is also proposed an original interpretation of the limitations due to diffraction observed in wave field propagation in terms of the different waves generated inside the cavity. The time-reversal focusing process using a closed cavity in a weakly inhomogeneous medium is finally compared with more classical techniques to compensate wavefront distortions, thus illustrating the focusing improvement due to the time-reversal method.**

I. Introduction

FOR SEVERAL YEARS, one of the most significant problems in propagation of acoustic and/or optic waves lies in wavefront or phase distortion correction. As an immediate and very classical example, the focusing of an ultrasonic pressure field on a specific target is a delicate operation as soon as the propagation medium presents inhomogeneities in compressibility and/or density. In such a situation, the interaction of the acoustic field with the inhomogeneities generates a distortion of the wavefront; it results therefore that the spatial localization of the target is not enough to optimize focusing.

Different techniques have been proposed to solve this problem as cross-correlation methods [19] and time-reversal mirrors [2]–[4]. Some experimental results obtained with time-reversal mirrors illustrate the ability of this technique to compensate the distortions introduced by several kinds of weak aberrating media [2]–[4], [19], [20]. If we are interested in focusing in a strongly inhomogeneous medium, the time-reversal mirror presents many efficiency limitations. Indeed, in such a situation, there are forward- and backward-scattering effects, while the time-reversal mirror is not able to measure all these components of the wave field. As an elementary example, we can consider the propagation of an incident wave along an interface separating two different media, thus generating a reflected wave and also a transmitted wave. A single time-reversal mirror can only measure and time-reverse one of these two waves, while a complete time-reversal process working simultaneously on the reflected and transmitted waves should be necessary to regenerate the initial incident wave.

The basic theory of the time-reversal process is based on an elementary property of the wave equation in a lossless medium. In the most general case, the wave equation is a differential equation where the time-derivative operator appears only at the second order. It results therefore that, if $p(\vec{r}, t)$ is a solution, then $p(\vec{r}, -t)$ is also a solution. In other words, the wave equation is unchanged under a time-reversal transform if there is no absorption during propagation in the medium. Now, if $p(\vec{r}, t)$ is the pressure field that propagates from a single source, the optimal way to focus on this source without knowing its absolute position consists in a time-reversal of $p(\vec{r}, t)$ in the whole 3-D volume, thus generating $p(\vec{r}, -t)$. In fact, we must consider a more realistic situation using the Huygens's principle, therefore reducing the time-reversal operation on a 3-D volume to a two-dimensional (2-D) surface. Starting from this observation, we have developed the concept of closed time-reversal cavity that works as a two-step process.

During the first or recording step, the target is considered as a source term that generates a pressure field $p(\vec{r}, t)$. We suppose that we are able to measure at any point on the surface of the cavity the pressure field and its normal derivative across the surface. We also suppose that the cavity does not perturb the propagation of the pressure field, that therefore propagates as in a free unbounded medium. During the second or reconstruction step, the initial source is removed or remains passive, and we suppose that we are able to create secondary sources on the surface of the cavity (monopole and dipole sources) that exactly correspond to the time-reversal of the corresponding field components measured during the first step. Our objective consists of computing the time-reversed pressure field that propagates inside the cavity according to the new boundary conditions on its surface. As it will be discussed in a forward section, such a time-reversal cavity cannot be realized experimentally. Nevertheless, it is still important to analyze it because the concept of cavity can be understood as theoretical limitation of the basic principle of the time-reversal.

In a first section, we consider the general problem of the closed time-reversal cavity process and obtain a complete expression of the time-reversed pressure field in the cavity. In the second part, we are interested in the particular case of a homogeneous propagation medium and give an interpretation of the self-focusing limitations in terms of the two spherical

Manuscript received September 30, 1991; revised February 21, 1992; accepted February 25, 1992.
The authors are with the Laboratoire Ondes et Acoustique, Université PARIS VII—ESPCI 10 rue Vauquelin, 75231 Paris Cedex 05, France.
IEEE Log Number 9201924.

waves generated by the cavity that respectively converge to and diverge from the initial source position. In the third part, the obtained results are extended to the propagation of ultrasonic fields in an inhomogeneous medium. Some similar conclusions can be given as in the case of a homogeneous medium, except that the focusing limitations can be now interpreted in terms of loss of information during the recording step. Finally, in a fourth part, we develop a generalization of the First Born Approximation to transient pulsed signals and analyze the obtained results in the particular case of a weakly inhomogeneous medium. The time-reversal process is compared with more classical focusing techniques, thus illustrating the focusing improvement due to the method.

The theoretical model we propose is valid in the transient regime; it includes and generalizes the monochromatic formalism used in optics as in the phase conjugation concept.

As mentioned in a previous paragraph, the time-reversal process requires that the wave equation remains unchanged under a time-reversal transform $t \Longrightarrow -t$. This condition cannot be satisfied in an absorbing medium since in this case, the wave equation presents a time-derivative operator of the first order. That is the reason why we consider in all of the following paper a lossless propagation medium. Experimentally, it can be observed in biological media that the time-reversal process works if the absorption coefficient is weak enough in the frequency range used in the experiment.

It is important to mention that this work is concerned in an optimal focalization of ultrasonic waves through aberrating media such as biological tissues. The most important applications of these concepts can be found in medical therapy as lithotripsy and hyperthermia. Using the same concepts in medical imaging appears to be difficult since the inverse problem we solve in this paper requires twice a propagation of ultrasonic waves in the inhomogeneous medium. As a consequence, the image of the medium is only available in the medium itself, and cannot be observed by any set of transducers. Under this condition, this study cannot be directly compared to computed tomography of inverse backscattering techniques, that generate the image of the medium through signal analysis and softwave algorithms.

II. Basic Theory of the Closed Time-Reversal Cavity

In the first or recording step of the self-focusing process with a closed time-reversal cavity, we consider an object source arbitrarily located at the origin of spatial coordinates. This source generates a scalar pressure field $p(\vec{r}, t)$ that satisfies the wave equation in the transient regime:

$$(\nabla^2 - c^{-2}\partial_{tt})p(\vec{r}, t) = -\phi(t)\delta(\vec{r}) - \mathcal{A}(\vec{r})\{p(\vec{r}, t)\} \quad (1)$$

where ∂_{tt} is the operator defined by $\partial_{tt} = (\partial^2/(\partial t^2))$, and c is the sound speed. In all of the following, ∇^2 represents the laplacian operator with respect to the $\vec{r}$-coordinates, while ∇_0^2 represents the laplacian operator with respect to the $\vec{r}_0$-coordinates. $\delta(\vec{r})$ represents the Dirac distribution in the three-dimensional (3-D) space and $\phi(t)$ is any function of time; it describes the temporal variations of the source excitation corresponding to the first term on the right-hand side of (1). The second term $\mathcal{A}(\vec{r})\{p(\vec{r}, t)\}$ has been introduced to take into account some local inhomogeneities in the propagation medium. This term is written as a formal linear operator $\mathcal{A}(\vec{r})$ working on the pressure field $p(\vec{r}, t)$. It will be shown in the third section of this paper how this description of the second term of (1) can be linked to the inhomogeneities in compressibility and/or density of the propagation medium. [1] Looking at the right-hand side of (1), the term corresponding to the inhomogeneities can be considered as a secondary source term since it represents sources of scattered sound produced by the interactions between the sound wave $p(\vec{r}, t)$ and the inhomogeneities; but it does not represent any new energy being introduced into the sound field.

The two following assumptions are made about the excitation function $\phi(t)$:

1) it is a causal function, i.e., $\phi(t)$ is zero for any time $t < 0$, and
2) it is defined on a finite support $[0, T_\phi]$, i.e., $\phi(t)$ is zero for any time $t > T_\phi$.

The first assumption is not restricting since any physical process is causal. With the second assumption, we consider that the temporal excitation of the source only exists on a finite time-interval; this is the case in almost all experimental situations.

If the pressure field propagates in a free unbounded medium, the solution to (1) can be found as [1]

$$p(\vec{r}, t) = \int_{\mathcal{V}} [\delta(\vec{r}_0)\phi(t) + \mathcal{A}(\vec{r}_0)\{p(\vec{r}_0, t)\}] \overset{*}{_t} G_d(\vec{r}, \vec{r}_0, t)\; d^3\vec{r}_0 \quad (2)$$

where $\mathcal{V}$ is a volume that contains all the sources (primary and secondary sources due to inhomogeneities) and $*/t$ is the time-convolution operator. $G_d(\vec{r}, \vec{r}_0, t)$ is the free-space Green's function which corresponds to an impulse diverging spherical wave and satisfies the propagation equation

$$(\nabla_0^2 - c^{-2}\partial_{tt})G_d(\vec{r}, \vec{r}_0, t) = -\delta(t)\delta(\vec{r} - \vec{r}_0). \quad (3)$$

A closed form solution to (3) is classically given by

$$G_d(\vec{r}, \vec{r}_0, t) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|}\delta\left(t - \frac{|\vec{r} - \vec{r}_0|}{c}\right). \quad (4)$$

We also define the impulse converging spherical wave:

$$G_c(\vec{r}, \vec{r}_0, t) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|}\delta\left(t + \frac{|\vec{r} - \vec{r}_0|}{c}\right)$$

which satisfies the same propagation equation as the impulse diverging spherical wave

$$(\nabla_0^2 - c^{-2}\partial_{tt})G_c(\vec{r}, \vec{r}_0, t) = -\delta(t)\delta(\vec{r} - \vec{r}_0).$$

Starting with the expression of the free-space Green's function given in (4), the pressure field $p(\vec{r}, t)$ given by (2) can be reduced to

$$p(\vec{r}, t) = \frac{1}{4\pi|\vec{r}|}\phi\left(t - \frac{|\vec{r}|}{c}\right) + \int_{\mathcal{V}} [\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0, t)\}] \overset{*}{_t} G_d(\vec{r}, \vec{r}_0, t) d^3\vec{r}_0. \quad (5)$$

The recording surface $\mathcal{S}$ is a closed cavity surrounding the object source and the whole volume $\mathcal{V}$ containing the inhomogeneities. We suppose here that the inhomogeneities are located in a bounded region of space, and that the surface of the cavity completely surrounds this region. The cavity is assumed to not perturb the propagation of the pressure field, such that the infinite free-space hypothesis remains valid.

Since the pressure field $p(\vec{r}, t)$ is generally not defined on a finite time support, we suppose that we are able to measure the pressure field and its normal derivative at any point of the surface on the cavity, but only during a finite time-interval $[0, T]$. Defining

$$W(t) = \begin{cases} 1, & \text{if } 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}$$

The recorded pressure field at any point of the surface of the cavity can be written as $p(\vec{r}, t)W(t)$. In order to better understand the first step of the process, we can imagine an infinite set of elementary transducers on the surface of the cavity that measure the pressure field and its normal derivative without perturbation with respect to the infinite free-space hypothesis. This first step is illustrated on Fig. 1 [5], [6]. It is important to note that the closed time-reversal cavity cannot be realized experimentally because:

1) first, an ultrasonic device working in the receiving mode does not measure simultaneously the pressure field and its normal derivative; the electrical output signal depends generally on these two quantities with relative weight coefficients that are specified in addition to the transducer,
2) second, it is not possible to impose specific boundary conditions in the emitting mode; the most commonly used approximation for computations is the piston mode hypothesis,
3) third, it is not possible to measure the pressure field and/or its normal derivative at any point of the surface of the cavity since the ultrasonic transducers have a finite aperture, therefore resulting in a spatial filtering of the incident acoustic field, and an information loss,
4) fourth, it is generally not easy to have an ultrasonic device that completely surrounds the region of interest; in any common situation, the emitting and receiving system is located on a single side of the region of interest.

In fact, the concept of closed time-reversal cavity can be understood as a theoretical extension of acoustic mirrors experimentally realized in the laboratory and presented in previous publications [2]–[4].

In the second step, we suppose that we are able to create secondary sources on the surface of the cavity (monopole and dipole sources) such that the boundary conditions on the surface $\mathcal{S}$ exactly correspond to the time-reversed components of the pressure field recorded during the first step. It results from the definition of $W(t)$ that the recorded field vanishes for any observation time $t > T$. As a consequence, and to insure causality, the time-reversal process can be described by

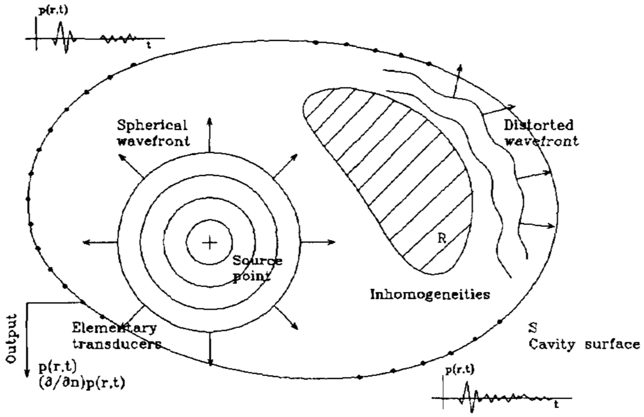

Fig. 1. The first or recording step of the time-reversal process with a closed cavity. A point-like source located at the origin generates a spherical wavefront that is distorted after propagation through the inhomogeneities of the medium. The closed time-reversal cavity surrounds the initial object source and the inhomogeneities. An infinite set of elementary transducers, located on the surface of the cavity measure the acoustic pressure and its normal derivative across the surface.

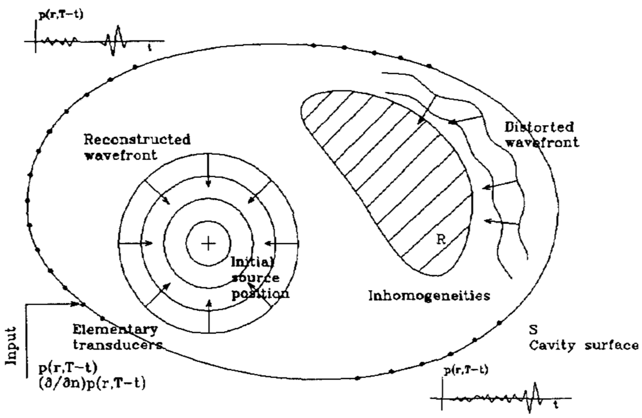

Fig. 2. The second or reconstruction step of the time-reversal process with a closed cavity. The initial source is removed or remains passive and the in homogeneities are unchanged. Each elementary transducer on the surface of the cavity generates the time-reversal of the pressure and its normal derivative measured during the first step. A time-reversed pressure field back-propagated inside the cavity and should be focused on the initial source position.

the transformation

$$t \Longrightarrow T - t.$$

It therefore results that the surface sources created on the surface of the cavity can be written as

$$\begin{cases} \sigma_1(\vec{r}_0, t) = p(\vec{r}_0, T - t)W(T - t) \\ \sigma_0(\vec{r}_0, t) = \vec{n}_0 \cdot \vec{\nabla}_0 p(\vec{r}_0, T - t)W(T - t) \end{cases} \tag{6}$$

where $\vec{n}_0$ is the normal vector to $\mathcal{S}$ oriented outward (*away from the cavity*), $\vec{r}_0$ represents any point on the surface of the cavity and $\vec{n}_0 \cdot \vec{\nabla}_0$ is the normal derivative operator. The two functions σ_1 and σ_0 correspond to the discontinuity of the pressure field (σ_1) and its normal derivative (σ_0) generated during the second step across the surface of the cavity. This second step is illustrated in Fig. 2.

The object source considered during the first or recording step *is now removed or remains passive; the inhomogeneities are unchanged.* It results from the new boundary conditions on the surface $\mathcal{S}$ of the cavity that a time-reversed pressure field, $p_{tr}(\vec{r}, t)$, back-propagates in the whole 3-D volume. Similarly to (1), the time-reversed pressure field satisfies the wave equation:

$$(\nabla^2 - c^{-2}\partial_{tt})p_{tr}(\vec{r}, t) = -\mathcal{A}(\vec{r})\{p_{tr}(\vec{r}, t)\} \tag{7}$$

where the second term on the right-hand side of (7) also results from the interaction of the time-reversed pressure field with the inhomogeneities.

The expression of the time-reversed pressure field inside the cavity can be obtained by a generalization of the Kirchhoff's formula: [1]

$$\begin{aligned} p_{tr}(\vec{r}, t) = &\int_{\mathcal{V}} [\mathcal{A}(\vec{r}_0)\{p_{tr}(\vec{r}_0, t)\}] \overset{*}{\underset{t}{}} G_d(\vec{r}, \vec{r}_0, t) d^3\vec{r}_0 + \\ &\int_{\mathcal{S}} \Big[\sigma_0(\vec{r}_0, t) \overset{*}{\underset{t}{}} G_d(\vec{r}, \vec{r}_0, t) \\ &\quad - \sigma_1(\vec{r}_0, t) \overset{*}{\underset{t}{}} \vec{n}_0 \cdot \vec{\nabla}_0 G_d(\vec{r}, \vec{r}_0, t)\Big] d^2\vec{r}_0. \end{aligned} \tag{8}$$

Using Green's theorem, the surface integral over $\mathcal{S}$ can be changed to a volume integral over the volume $\mathcal{V}$ enclosed by $\mathcal{S}$ (inside of the cavity), such that (8) can be written in another form [7]:

$$\begin{aligned} p_{tr}(\vec{r}, t) = &\int_{\mathcal{V}} [\mathcal{A}(\vec{r}_0)\{p_{tr}(\vec{r}_0, t)\}] \overset{*}{\underset{t}{}} G_d(\vec{r}, \vec{r}_0, t) d^3\vec{r}_0 + \\ &\int_{\mathcal{V}} \Big[G_d(\vec{r}, \vec{r}_0, t) \overset{*}{\underset{t}{}} \nabla_0^2 \sigma_1(\vec{r}_0, t) \\ &\quad - \sigma_1(\vec{r}_0, t) \overset{*}{\underset{t}{}} \nabla_0^2 G_d(\vec{r}, \vec{r}_0, t)\Big] d^3\vec{r}_0 \end{aligned} \tag{9}$$

where $\sigma_1(\vec{r}_0, t)$ is a function of time resulting from a truncation of the pressure field $p(\vec{r}_0, T-t)$ through the temporal window $W(T - t)$; it results therefore that this function generally presents discontinuities, such that the temporal derivation must be evaluated in the frame of the distribution theory. Starting from this remark and from (1) and (6), we can verify that we have the two following relations:

$$\begin{cases} (\nabla_0^2 - c^{-2}\partial_{tt})p(\vec{r}_0, T - t) = -\phi(T - t)\delta(\vec{r}_0) \\ \qquad\qquad -\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0, T - t)\} \\ \partial_{tt}\sigma_1(\vec{r}_0, t) = W(T - t)\partial_{tt}p(\vec{r}_0, T - t) \\ \qquad\qquad +\varepsilon(\vec{r}_0)\delta'(t) - \eta(\vec{r}_0)\delta(t) \end{cases} \tag{10}$$

where $\delta'(t)$ is the temporal derivative of the Dirac distribution $\delta(t)$ and $\varepsilon(\vec{r}_0)$ and $\eta(\vec{r}_0)$ represent the discontinuities of the pressure field $p(\vec{r}_0, t)$ and its temporal derivative at $t = T$ (note that the pressure field and its normal derivative both equate zero at $t = 0$ for causality reasons). These two functions of space are given by

$$\varepsilon(\vec{r}_0) = p(\vec{r}_0, t)\Big|_{t=T} \text{ and } \eta(\vec{r}_0) = \frac{\partial}{\partial t}p(\vec{r}_0, t)\Big|_{t=T}. \tag{11}$$

It is shown in Appendix I that the time-reversed pressure field $p_{tr}(\vec{r}, t)$ can be reduced to (12), shown at the bottom of the page.

We are now interested in the choice of the recording time-interval $[0, T]$. Defining d_m as the greatest distance between the initial object source position and the different points of the surface of the cavity $\mathcal{S}$:

$$d_m = \max_{\vec{r} \in \mathcal{S}}(|\vec{r}|).$$

It is clear that the recording duration T should be at least equal to $(T_\phi + d_m/c)$, that corresponds to the minimum observation time-interval to measure the complete pressure field in the case of a homogeneous medium (d_m/c is the maximum time of propagation from the source position to any observation point on the surface of the cavity, and T_ϕ corresponds to the temporal duration of the excitation function $\phi(t)$). In the case of an inhomogeneous medium, T should be chosen greater than this value $(T_\phi + d_m/c)$. The particular choice of T, depending on the kind of inhomogeneities, will be discussed in a forward section.

Considering that the parameter T is chosen such that the above condition is satisfied, we can verify that the first two terms of (12) can be reduced to

$$\begin{cases} \frac{1}{4\pi|\vec{r}|}\phi\left(T - t - \frac{|\vec{r}|}{c}\right)W(T - t) \\ \qquad\qquad \equiv \frac{1}{4\pi|\vec{r}|}\phi\left(T - t - \frac{|\vec{r}|}{c}\right) \\ \frac{1}{4\pi|\vec{r}|}\phi\left(T - t + \frac{|\vec{r}|}{c}\right)W\left(T - t + \frac{|\vec{r}|}{c}\right) \\ \qquad\qquad \equiv \frac{1}{4\pi|\vec{r}|}\phi\left(T - t + \frac{|\vec{r}|}{c}\right). \end{cases} \tag{13}$$

$$\begin{aligned} p_{tr}(\vec{r}, t) = &\frac{1}{4\pi|\vec{r}|}\phi\left(T - t - \frac{|\vec{r}|}{c}\right)W(T - t) - \frac{1}{4\pi|\vec{r}|}\phi\left(T - t + \frac{|\vec{r}|}{c}\right)W\left(T - t + \frac{|\vec{r}|}{c}\right) \\ &+ W(T - t)\int_{\mathcal{V}} [\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0, T - t)\}] \overset{*}{\underset{t}{}} G_c(\vec{r}, \vec{r}_0, t) d^3\vec{r}_0 \\ &- \int_{\mathcal{V}} [W(T - t)\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0, T - t)\}] \overset{*}{\underset{t}{}} G_d(\vec{r}, \vec{r}_0, t) d^3\vec{r}_0 \\ &+ \int_{\mathcal{V}} [\mathcal{A}(\vec{r}_0)\{p_{tr}(\vec{r}_0, t)\}] \overset{*}{\underset{t}{}} G_d(\vec{r}, \vec{r}_0, t) d^3\vec{r}_0 \\ &+ \frac{1}{c^2}\int_{\mathcal{V}} \left[\eta(\vec{r}_0)\delta\left(t - \frac{|\vec{r} - \vec{r}_0|}{c}\right) - \varepsilon(\vec{r}_0)\delta'\left(t - \frac{|\vec{r} - \vec{r}_0|}{c}\right)\right] \frac{d^3\vec{r}_0}{4\pi|\vec{r} - \vec{r}_0|}. \end{aligned} \tag{12}$$

III. Self-Focusing in a Homogeneous Medium

We now consider the same problem as in the previous section, except that the medium is homogeneous. In such case, the two wave equations given by (1) and (7) remain valid with

$$\mathcal{A}(\vec{r}) = 0.$$

for any value of $\vec{r}$. If the recording duration T is chosen according to the preceding part, the pressure field and its temporal derivative vanish for any observation time $t \geq T$ inside the cavity (the pressure field propagates in a free unbounded homogeneous medium). As a consequence, we have $\varepsilon(\vec{r}) = 0$ and $\eta(\vec{r}) = 0$. for any observation point $\vec{r}$ inside the cavity. It results therefore that the time-reversed pressure field $p_{tr}(\vec{r}, t)$ given by (12) reduces to a very simple form [5], [6]:

$$p_{tr}(\vec{r}, t) = \frac{1}{4\pi|\vec{r}|}\phi\left(T - t - \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi|\vec{r}|}\phi\left(T - t + \frac{|\vec{r}|}{c}\right). \tag{14}$$

Introducing the kernel distribution

$$K(\vec{r}, t) = \frac{1}{4\pi|\vec{r}|}\delta\left(t + \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi|\vec{r}|}\delta\left(t - \frac{|\vec{r}|}{c}\right). \tag{15}$$

Equation (14) can be rewritten as

$$p_{tr}(\vec{r}, t) = \phi(T - t) \underset{t}{\overset{*}{}} K(\vec{r}, t). \tag{16}$$

This last expression of the time-reversed pressure field is very simple and allows us to give some physical interpretation of what happens during the self-focusing process inside the cavity. First, it is interesting and important to note that the time-reversed pressure field $p_{tr}(\vec{r}, t)$ varies linearly with respect to $\phi(T - t)$ (in the sense of the linear system theory), and not with respect to $\phi(t)$. This remark is important if the self-focusing process has to be investigated in the frequency domain instead of the time domain. This result is not surprising since the transformation realized on the surface of the cavity has time-reversed all the components of the pressure field generated during the first step.

The kernel distribution $K(\vec{r}, t)$ corresponds to the difference of two impulse spherical waves that respectively converge to and diverge from the origin of spatial coordinates, i.e., the initial source position. It results therefore that the self-focusing process with a closed cavity leads to a time-reversed pressure field $p_{tr}(\vec{r}, t)$ that is not perfectly focused on the initial object source [5].

Although both spherical waves show a singularity at the origin, it is interesting to note that the time-reversed pressure field has a finite value for any t and $\vec{r}$. This result can be explained in the frame of the general theory for wave propagation. Indeed, during the second or reconstruction step, the initial source is removed or remains passive. As a consequence, there is no more spatial discontinuity with respect to the acoustic field that propagates inside the cavity, and the pressure field resulting from this back-propagation cannot be discontinuous on the inside of the volume $\mathcal{V}$. An immediate consequence of the preceding results is that the time-reversed pressure field at the origin is given by

$$p_{tr}(\vec{0}, t) = -\frac{1}{2\pi c}\phi'(T - t)$$

which results in a temporal derivation of the excitation function [5] (in this expression, $\phi'(t)$ is defined as the temporal derivative of the excitation function $\phi(t)$).

The time-reversed pressure field, observed as a function of time, shows two different wavefronts. The second wavefront is the exact replica of the first one, except that it is time-reversed and multiplied by -1. The arrival-time difference between the two wavefronts is $2|\vec{r}|/c$, this difference increases with the distance separating the observation point and the initial source position. If this difference is greater that T_ϕ (the observation point is located *far from the origin*), the two wavefronts can be separated in time. Otherwise (for an observation point located *near the origin*), the two wavefronts overlap, therefore resulting in a distortion of the temporal variations of the time-reversed pressure field. This distortion effect, caused by overlapping of the two wavefronts, leads to the previously mentioned temporal derivation of the excitation function in the neighborhood of the origin.

The superposition of the two spherical waves, converging to and diverging from the origin, is directly related to the theoretical focusing limitations (in terms of loss of resolution) of the process. Taking the Fourier transform of (16) over the time variable t, we obtain after some elementary computations

$$\tilde{p}_{tr}(\vec{r}, \omega) = \tilde{K}(\vec{r}, \omega)\tilde{\phi}(\omega)^* \exp(j\omega T) \tag{17}$$

with

$$\tilde{K}(\vec{r}, \omega) = \frac{1}{2j\pi}\frac{\sin(\omega|\vec{r}|/c)}{|\vec{r}|} = \frac{1}{j\lambda}\frac{\sin(k|\vec{r}|)}{k|\vec{r}|} \tag{18}$$

where λ is the wavelength defined by $\lambda = 2\pi c/\omega$ and k is the wave number given by $k = 2\pi/\lambda = \omega/c$. In this equation, we recognize the expression of a *spherical Bessel function.*

Looking at the preceding expressions for the time-reversed pressure field and its Fourier transform over the time variable t, we can make the following remarks:

1) the obtained result does not depend on the shape of the surface $\mathcal{S}$, i.e., any time-reversal closed cavity gives the same image of a point source,
2) the time-reversed pressure field has the spherical symmetry,
3) in the frequency domain, the maximum available resolution for the self-focusing process with a closed cavity is $\lambda/2$; we retrieve here a classical result of optical holography in the frame of the so-called basic image system. [9]–[12]

IV. Self-Focusing in an Inhomogeneous Medium

We are now interested in an extension of the preceding self-focusing process to the reconstruction of a point-like source located in an inhomogeneous medium. The inhomogeneities correspond to spatial variations of the compressibility and/or density of the medium; they are assumed to be located in a specific region $\mathcal{R}$ of space that can be inscribed in a finite volume near the origin.

Inside $\mathcal{R}$, the compressibility variations of the medium are described through a function $\kappa_i(\vec{r})$, while the density variations are described through $\rho_i(\vec{r})$. Outside $\mathcal{R}$, the propagation medium is homogeneous with a compressibility κ and a density ρ. In all of the following, we prefer to describe the inhomogeneities of the propagation medium through the relative variations of compressibility and density with respect to κ and ρ. We introduce the following notations:

$$\begin{cases} \gamma_\kappa(\vec{r}) = \frac{\kappa_i(\vec{r})-\kappa}{\kappa} \quad \text{and} \quad \gamma_\rho(\vec{r}) = \frac{\rho_i(\vec{r})-\rho}{\rho_i(\vec{r})}, & \text{inside } \mathcal{R} \\ \gamma_\kappa(\vec{r}) = \gamma_\rho(\vec{r}) = 0, & \text{outside } \mathcal{R}. \end{cases}$$

We finally suppose that the inhomogeneities are a function of only the spatial coordinates, and do not depend on time. Since the compressibility and density parameters result from physical properties specific to the propagation medium, the two functions $\gamma_\kappa(\vec{r})$ and $\gamma_\rho(\vec{r})$ have real values. The last assumption is that the cavity contains the whole region of space $\mathcal{R}$ where the propagation medium shows local inhomogeneities.

The propagation equation can be derived from the general equations of acoustics, taking into account the spatial variations of compressibility and density of the medium. This equation can be found in the transient regime as [1]

$$(\nabla^2 - c^{-2}\partial_{tt})p(\vec{r},t) = c^{-2}\gamma_\kappa(\vec{r})\partial_{tt}p(\vec{r},t) + \vec{\nabla}\cdot\left[\gamma_\rho(\vec{r})\vec{\nabla}p(\vec{r},t)\right] \quad (19)$$

where the propagation velocity c is defined by $c = (\rho\kappa)^{-1/2}$. If we consider now a point-like source located at the origin, (19) remains valid, but it is necessary to introduce a supplementary term like $-\phi(t)\delta(\vec{r})$ on the right-hand side of the equation, similar to (1). Looking at (19), it is clear that the wave equation has the same structure as (1) if we define the formal operator $\mathcal{A}(\vec{r})$ by

$$\mathcal{A}(\vec{r}) = -c^{-2}\gamma_\kappa(\vec{r})\partial_{tt} - \vec{\nabla}\cdot\left[\gamma_\rho(\vec{r})\vec{\nabla}\right]. \quad (20)$$

As a consequence, the time-reversed pressure field generated in the case of an inhomogeneous propagation medium results from an immediate application of (12).

The time-reversed pressure field given in (12) is not simple since it requires the evaluation of several integrals over the volume where the propagation medium is inhomogeneous. The evaluation of these integrals is a delicate operation since (12) is self-consistent; the knowledge of the time-reversed pressure field $p_{tr}(\vec{r},t)$ requires the previous evaluation of $p(\vec{r},t)$ and $p_{tr}(\vec{r},t)$ itself.

In order to provide a more simple interpretation of the obtained results, we do not consider the time-reversed pressure field $p_{tr}(\vec{r}_0,t)$, but we prefer to define

$$P(\vec{r}_0,t) = p(\vec{r}_0,T-t)W(T-t) - p_{tr}(\vec{r}_0,t).$$

It results from this definition and (10) that we have

$$\begin{aligned}(\nabla_0^2 - c^{-2}\partial_{tt})P(\vec{r}_0,t) = &-\phi(T-t)W(T-t)\delta(\vec{r}_0) \\ &- \varepsilon(\vec{r}_0)\delta'(t)/c^2 + \eta(\vec{r}_0)\delta(t)/c^2 \\ &- W(T-t)\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0,T-t)\} \\ &+ \mathcal{A}(\vec{r}_0)\{p_{tr}(\vec{r}_0,t)\}. \quad (21)\end{aligned}$$

Starting from this expression, we are going to prove that $P(\vec{r}_0,t)$ also satisfies a self-consistent equation. To do that, we introduce in (21) a term like $\mathcal{A}(\vec{r}_0)\{W(T-t)p(\vec{r}_0,T-t)\}$ that, combined with the last term $\mathcal{A}(\vec{r}_0)\{p_{tr}(\vec{r}_0,t)\}$, exactly reduces to the self-consistent term. The problem is now to evaluate the difference between the desired term and the one effectively present in (21):

$$\begin{aligned}D(\vec{r}_0,t) = &\mathcal{A}(\vec{r}_0)\{W(T-t)p(\vec{r}_0,T-t)\} \\ &- W(T-t)\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0,T-t)\}. \quad (22)\end{aligned}$$

The evaluation of this difference refers to the specific form of the operator $\mathcal{A}(\vec{r}_0)$. As written in (20), the operator $\mathcal{A}(\vec{r}_0)$ shows two terms, the first one operating on time $(-c^{-2}\gamma_\kappa(\vec{r}_0)\partial_{tt})$ and the second one only on spatial variables $\left(-\vec{\nabla}\cdot\left[\gamma_\rho(\vec{r}_0)\vec{\nabla}\right]\right)$. Since the second term does not affect the time variable, it is clear that the corresponding difference coming from (22) vanishes. Therefore, the difference term $D(\vec{r}_0,t)$ is directly related to the difference between $\partial_{tt}\{W(T-t)p(\vec{r}_0,T-t)\}$ and $W(T-t)\partial_{tt}\{p(\vec{r}_0,T-t)\}$. Starting from this remark and using (10), we immediately obtain

$$D(\vec{r}_0,t) = -\gamma_\kappa(\vec{r}_0)[\varepsilon(\vec{r}_0)\delta'(t) - \eta(\vec{r}_0)\delta(t)]. \quad (23)$$

Finally introducing this expression in (21) and considering the linearity of the operator $\mathcal{A}(\vec{r}_0)$, we obtain the following equation for $P(\vec{r}_0,t)$:

$$\begin{aligned}(\nabla_0^2 - c^{-2}\partial_{tt})P(\vec{r}_0,t) = &\\ &-\phi(T-t)W(T-t)\delta(\vec{r}_0) \\ &- \mathcal{A}(\vec{r}_0)\{P(\vec{r}_0,t)\} \\ &- \frac{1+\gamma_\kappa(\vec{r}_0)}{c^2}[\varepsilon(\vec{r}_0)\delta'(t) - \eta(\vec{r}_0)\delta(t)]. \quad (24)\end{aligned}$$

This expression is interesting since it means that $P(\vec{r}_0,t)$ satisfies a self-consistent equation with a source term given by $-\phi(T-t)W(T-t)\delta(\vec{r}_0)$ and a perturbation term due to the discontinuity of the pressure field and its temporal derivative measured during the first step of the process. It is important to note that this perturbation term does not depend on the inhomogeneities in density ($\gamma_\rho(\vec{r}_0)$); this term is not only due to inhomogeneities since it does not vanish in the case of a homogeneous propagation medium. An immediate interpretation of this correction term resides in the fact that some information coming from the source throughout the medium is lost due to the time-window $W(t)$ during the first or recording step, therefore resulting in a perturbation of the optimal focusing. This correction term naturally vanishes if the information has been completely measured (this is the case considered in the previous section with the adequate choice of the parameter T); and that the resulting perturbation increases with the loss of information.

In order to simplify the next developments, we first suppose that the correction term previously mentioned equates zero or is completely negligible, such that the time-windowing has no effect on the pressure field measurement and/or reconstruction. We also introduce $h(\vec{r},t)$ as the solution of (1) in the impulse regime ($\phi(t) = \delta(t)$). It results from general linear system theory that $p(\vec{r}, t)$ and $P(\vec{r}, t)$ can be respectively rewritten as

$$\begin{cases} p(\vec{r}, t) = \phi(t) \overset{*}{_t} h(\vec{r}, t) \\ P(\vec{r}, t) = \phi(T - t) \overset{*}{_t} h(\vec{r}, t). \end{cases} \tag{25}$$

It results now from the definition of $P(\vec{r}, t)$ and from (25) that the time-reversed pressure field $p_{tr}(\vec{r}, t)$ reduces to a very simple form

$$\begin{aligned} p_{tr}(\vec{r}, t) &= \underbrace{\phi(T - t) \overset{*}{_t} h(\vec{r}, -t)}_{p(\vec{r}, T-t)} - \underbrace{\phi(T - t) \overset{*}{_t} h(\vec{r}, t)}_{P(\vec{r}, t)} \\ &= \phi(T - t) \overset{*}{_t} [h(\vec{r}, -t) - h(\vec{r}, t)]. \end{aligned} \tag{26}$$

This last expression of the time-reversed pressure field $p_{tr}(\vec{r}, t)$ is very important and shows a similar structure as (15) and (16) previously obtained in the case of a homogeneous propagation medium. Indeed, in that specific case, the Green's solution $h(\vec{r}, t)$ reduces to an impulse diverging spherical wave, such that the difference $h(\vec{r}, -t) - h(\vec{r}, t)$ leads immediately to the definition of the kernel distribution $K(\vec{r}, t)$. Starting from this point of view, the obtained expression of the time-reversed pressure field $p_{tr}(\vec{r}, t)$ given by (26) results from a direct generalization of (16) for a homogeneous medium, where $h(\vec{r}, t)$ corresponds now to the Green's solution that takes into account the inhomogeneities of the propagation medium.

Similarly to the previous section, the time-reversed pressure field shows two different wavefronts. The first one exactly corresponds to the back-propagation of the pressure field $p(\vec{r}, t)$ generated during the first step, and the second one is the exact replica of the first one, time-reversed and multiplied by -1. Now we omit this second term and are interested in the first one $\phi(T - t) \overset{*}{_t} h(\vec{r}, -t)$ only.

Looking at $h(\vec{r}, t)$ as a function of time, it results from classical causality reasons that we first observe the effects of the diverging spherical wave radiated from the source, and then the interactions of this spherical wave (in fact of the total pressure field itself) with the inhomogeneities of the medium. Since the first term appearing in (26) exactly corresponds to the time-reversal of $h(\vec{r}, t)$, it results from the preceding remark that, during the second or reconstruction step, we first observe the effects of inhomogeneities (in the reverse order in comparison with the first step of the time-reversal process), and then an impulse spherical wave converging to the initial source position.

Also for the same causality reason, the second term $-h(\vec{r}, t)$ in (26) equates zero as long as the impulse spherical wave has not perfectly converged to the initial source position, thus generating a singularity of the reconstructed pressure field. Starting from this observation time, the first term $h(\vec{r}, -t)$ vanishes and the only second one exists, therefore regenerating the same events in the reverse order (or in the same order as during the first step of the process). Imagine we are able to create a video film of the propagation of the pressure field during the first step $p(\vec{r}, t)$; the reconstruction process can be understood as a projection of this video film in the reverse order, immediately followed by a re-projection in the initial order. It is important to note that the birth of the second wavefront immediately after the end of the first one avoids the singularity of the resulting pressure field at the origin, as explained in the previous section. It is also interesting to note that the two wavefronts are completely separated in time (except in the neighborhood of the origin) if we consider the Green's impulse solution $h(\vec{r}, -t) - h(\vec{r}, t)$; it results therefore that the different events can be separately and sequentially observed as time increases. As in the case of a homogeneous medium, an overlap effect of the two wavefronts can occur due to the convolution with the temporal signal shape $\phi(T - t)$, that naturally corresponds to the time-reversal of the initial shape $\phi(t)$.

The problem we would like to mention here consists now in taking into account the effects of the time-window $W(t)$ during the recording step. Considering a propagation equation like (1), the pressure field $p(\vec{r}, t)$ is described through a self-consistent equation. The closed form solution given in (5) can be interpreted as an incident acoustic beam (in our case, the incident beam is a spherical diverging wave) that interacts with the inhomogeneities through a volume integral. If we consider the point of view of a multi-scattering medium, it is clear that the pressure field $p(\vec{r}, t)$ decreases with t, but never vanishes (this is of course a theoretical consideration); that is the reason why we have introduced the time-window $W(t)$. Looking at (24), it appears that the correction term to add to (26) is directly related to the discontinuity of the pressure field ($\varepsilon(\vec{r})$) and its temporal derivative ($\eta(\vec{r})$). Although we do not try to give here a quantitative description of this correction term, the preceding remarks are interesting since they provide an elementary interpretation of this disturbance in terms of loss of information. In most practical cases, it is enough to choose the parameter T such that the pressure field $p(\vec{r}, t)$ and its temporal derivative are as negligible as possible at $t = T$; under this condition, the preceding interpretation of the time-reversed pressure field remains valid. The previous argument for choosing a value of T requires that the pressure field and its normal derivative vanish in the inside of the cavity, such that the choice of T clearly depends on the type of inhomogeneity (if the medium is weakly inhomogeneous, multiple scattering effects are negligible and the temporal variations of the pressure field decrease rapidly with time, while in the particular case of strong inhomogeneities, the temporal duration of the signals can be very large due to multiple scattering). In fact, it is difficult to obtain an optimal value of T that can be valid for any kind of inhomogeneous medium. Furthermore, a time-reversal experiment requires the storage of the different signals in shift registers [2]–[4], such that T must be chosen as short as possible to reduce the dynamic memory management and the cost of the hardware configuration of the experiment.

V. Application to a Weakly Inhomogeneous Medium

In this section, we are going to develop a specific application of the previous results in the case of a weakly inhomogeneous medium. In order to simplify the forward equations, we also suppose that the propagation medium shows inhomogeneities of compressibility only, and that the density does not vary with space ($\gamma_\rho(\vec{r}) = 0$).

As illustrated by the previous sections, an important difficulty resides in the propagation equations being self-consistent, such that we cannot obtain closed form solutions in a simple way without approximations. Since the propagation medium is supposed to be weakly inhomogeneous, it is possible to give an alternate form of (5) resulting from an extension in the time domain of the well known first-born approximation (FBA) [13]–[15].

The basic principle of the FBA consists in a description of the pressure field $p(\vec{r}, t)$ as a sum of two terms; the first one corresponds to an incident acoustic field and is preponderant, and the second one results from the interaction of the total pressure field with the inhomogeneities and is considered as a small correction term. Since this second term depends on the total pressure field itself, the FBA consists in replacing the total pressure field by the incident one. In such case, the incident acoustic field can be understood as the zero-order term, and the resulting volume integral over the inhomogeneities as the first-order term of a perturbation development of the exact solution [14]–[16]. The different terms of higher order are neglected. The problem is now to discuss the validity of this approximation in our situation.

It is well known that the FBA and its validity are frequency-dependant, since the approximation is only valid in the low-frequency range [13]–[15]. It results therefore that we cannot extend such an approximation in the impulse domain. In fact, considering the different expressions of the pressure field $p(\vec{r}, t)$ and $p_{tr}(\vec{r}, t)$, we can see that the evaluation of the temporal variations of the acoustic fields results from a convolution either with $\phi(t)$ or with $\phi(T - t)$. Practically, the temporal excitation function $\phi(t)$ works as a low-pass or band-pass filter. It is therefore enough to insure the validity of the FBA in the whole frequency range contained in the spectrum of $\phi(t)$. Under such conditions, using the FBA for the different frequencies and inverse Fourier transforming the obtained equations, it is possible to describe the pressure field in the time domain as the sum of a zero-order and a first-order term. In all of the following, we suppose that the above condition is satisfied. We keep the same notations as in the previous sections, but we add a "B" subscript to indicate that the corresponding quantities are evaluated in the frame of the extended FBA.

We are first interested in the pressure field $p(\vec{r}, t)$ and the corresponding Green's solution $h(\vec{r}, t)$ previously introduced. It results from the different assumptions concerning the inhomogeneities and the extended FBA that we can give a closed form and completely determined expression of $h_B(\vec{r}, t)$ as (27), shown at the bottom of the page where $\mathcal{D}(\vec{r}, \vec{r}_0) = |\vec{r}_0| + |\vec{r} - \vec{r}_0|$.

Looking at (27) and defining now d_m as the greatest distance between the different sources (the origin and the secondary sources of scattered sound due to the inhomogeneities) and the surface of the cavity $\mathcal{S}$,

$$d_m = \max_{\vec{r} \in \mathcal{S}, \vec{r}_0 \in \mathcal{V}} (|\vec{r}|, |\vec{r}_0| + |\vec{r} - \vec{r}_0|),$$

it is clear that the Green's solution $h_B(\vec{r}, t)$ vanishes at any point on the surface of the cavity $\mathcal{S}$ for any observation time $t > d_m/c$. Since $p_B(\vec{r}, t)$ results from the convolution of $\phi(t)$ with $h_B(\vec{r}, t)$, the pressure field on and inside the cavity equates zero for any value of $t > T_\phi + d_m/c$. As an immediate consequence, an adequate choice of the parameter T allows us to suppress the time-window $W(t)$ in all the equations, since in that case the whole information coming from the source throughout the medium is measured and time-reversed. Under this condition, the expression of the time-reversed pressure field given in (26) is valid. Similarly to (15) and (16), we introduce the generalized kernel distribution $K_B(\vec{r}, t)$ defined by

$$K_B(\vec{r}, t) = h_B(\vec{r}, -t) - h_B(\vec{r}, t).$$

It results immediately from this definition and from (27) the following expression of $K_B(\vec{r}, t)$ as (28), shown at the bottom of the page.

This last expression ((28)) is interesting since it can be easily interpreted geometrically. The first two Dirac distributions exactly correspond to the kernel distribution $K(\vec{r}, t)$ introduced in the case of a homogeneous medium. The correction term due to the extended FBA consists in a volume integral of Dirac distributions over the region of space inside the cavity that contains the inhomogeneities. If we consider a particular observation point $\vec{r}$ and an observation time $t > 0$, it results from the classical properties of the Dirac distributions that the only points of space that have a contribution to the correction term are such that $\gamma_\kappa(\vec{r}_0) \neq 0$ and $\mathcal{D}(\vec{r}, \vec{r}_0) = ct$. These two conditions describe a surface in the 3-D volume

$$\begin{aligned} h_B(\vec{r}, t) &= \frac{1}{4\pi|\vec{r}|}\delta\left(t - \frac{|\vec{r}|}{c}\right) + \int_{\mathcal{V}} \left[-c^{-2}\gamma_\kappa(\vec{r}_0)\partial_{tt}\frac{1}{4\pi|\vec{r}_0|}\delta\left(t - \frac{|\vec{r}_0|}{c}\right)\right] \underset{t}{*} G_d(\vec{r}, \vec{r}_0, t)d^3\vec{r}_0 \\ &= \frac{1}{4\pi|\vec{r}|}\delta\left(t - \frac{|\vec{r}|}{c}\right) - \frac{1}{16\pi^2c^2}\partial_{tt}\int_{\mathcal{V}} \frac{\gamma_\kappa(\vec{r}_0)}{|\vec{r}_0| \times |\vec{r} - \vec{r}_0|}\delta\left(t - \frac{\mathcal{D}(\vec{r}, \vec{r}_0)}{c}\right)d^3\vec{r}_0 \end{aligned} \tag{27}$$

$$\begin{aligned} K_B(\vec{r}, t) = &\frac{1}{4\pi|\vec{r}|}\delta\left(t + \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi|\vec{r}|}\delta\left(t - \frac{|\vec{r}|}{c}\right) - \frac{1}{16\pi^2c^2} \times \\ &\partial_{tt}\int_{\mathcal{V}} \frac{\gamma_\kappa(\vec{r}_0)}{|\vec{r}_0| \times |\vec{r} - \vec{r}_0|}\left[\delta\left(t + \frac{\mathcal{D}(\vec{r}, \vec{r}_0)}{c}\right) - \delta\left(t - \frac{\mathcal{D}(\vec{r}, \vec{r}_0)}{c}\right)\right]d^3\vec{r}_0 \end{aligned} \tag{28}$$

that has to be intersected with the volume $\mathcal{R}$ containing the inhomogeneities. If the observation point is located at the origin, this surface reduces to a sphere, centered at the origin, whose diameter ct increases with time. Otherwise, the surface is an ellipsoid with a symmetry axis given by the observation point position and the origin ($\vec{r}$ direction). The ellipsoid can be characterized by two foci $\vec{o}$ and $\vec{r}$, with a major axis and a minor axis respectively given by ct and $\sqrt{c^2t^2 - |\vec{r}|^2}$ (this relation implies that $ct \geq |\vec{r}|$, otherwise the surface reduces to the empty set). It is finally interesting to note that the shape of the ellipsoid changes when t increases, and looks like a sphere for large values of time.

Practically, if we want to compute the correction term of the kernel distribution $K_B(\vec{r}, t)$ for an observation point $\vec{r}$, we start from $t = 0$ and progressively increase the value of the observation time. The above surface grows with t (diameter of the sphere or major/minor axis of the ellipsoid) and is intersected with the volume $\mathcal{R}$. The resulting surface contains all the inhomogeneities that have a contribution to the correction term of the kernel distribution at $\vec{r}$ and t. The last step consists in the evaluation of the integral over this surface. This geometrical interpretation is illustrated on Fig. 3. The main interest of this description of the correction term is to interpret its value at an observation time t as coming from a very specific and limited region of $\mathcal{R}$; the correction term observed as a function of time contains a description of these different regions, that can be understood as different depths of inhomogeneities with respect to the observation point. This kind of description varies with the observation point, such that the complete information observed in the whole 3-D volume contains an image of the inhomogeneities. Finally, looking at (28), the correction term is an odd function of time.

Similarly to the case of a homogeneous propagation medium, we can compute the Fourier transform of the kernel distribution $K_B(\vec{r}, t)$ in order to provide an interpretation of the self-focusing process and justify the use of the FBA. It results immediately from (28) that the Fourier transform of $K_B(\vec{r}, t)$ is given by (29), shown at the bottom of the page.

The first term of this equation is exacly the same as in (18) corresponding to the particular case of a homogeneous propagation medium. The second term $\tilde{C}(\vec{r}, \omega)$ results from an integral over the volume containing the inhomogeneities; it can be understood as a correction term if the inhomogeneities are weak enough. Starting from (29), we can derive a simpler expression of this correction term if we consider an observation point $\vec{r}$ located in the neighborhood of the origin and inhomogeneities far enough from the origin. Indeed, in such

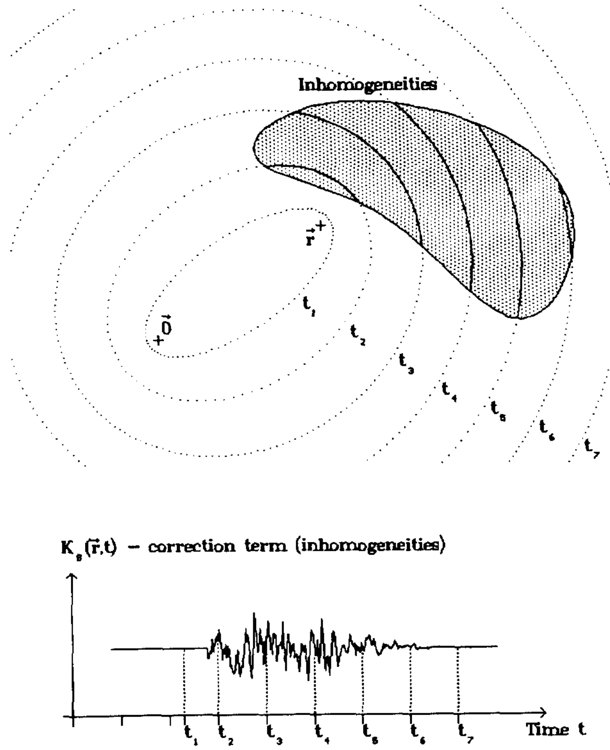

Fig. 3. Geometrical interpretation of the correction term due to inhomogeneities in the expression of $K_B(\vec{r}, t)$; $\vec{0}$ and $\vec{r}$, respectively, represent the origin of spatial coordinates (initial source position) and the observation point, the dot-filled area corresponds to a plane section of the volume $\mathcal{R}$ containing the inhomogeneities. the ellipsoid is represented at different observation times; it looks more and more to a sphere as time increases. The correction term results from an integration of the intersection of the ellipsoid with $\mathcal{R}$. For $t = t_1$ or $t = t_7$, this intersection reduces to the empty set, such that the resulting correction term is zero.

conditions, we can use the following approximations:

$$\begin{cases} |\vec{r} - \vec{r}_0| \simeq |\vec{r}_0| - \vec{r} \cdot \vec{u}_0, & (\vec{u}_0 \text{ being defined by } \vec{u}_0 = \vec{r}_0 / |\vec{r}_0|) \\ |\vec{r} - \vec{r}_0|^{-1} \simeq |\vec{r}_0|^{-1}. \end{cases}$$

If L_0 is the minimum distance between the origin and the volume $\mathcal{R}$ containing the inhomogeneities (see Fig. 4), we can verify that the previous approximations are only valid in a domain near the origin such that $|\vec{r}| \ll L_0$ and $|\vec{r}|^2 \ll \lambda L_0$ (the first condition is immediate and the second one results from the fact that the first neglected term in the sine function must be small with respect to 1). It results now from these approximations that the correction term given in (29) reduces to

$$\tilde{C}(\vec{r}, \omega) = \frac{\omega^2}{8j\pi^2c^2} \int_{\mathcal{V}} \frac{\gamma_\kappa(\vec{r}_0)}{|\vec{r}_0|^2} \sin\left(\frac{2\omega|\vec{r}_0|}{c} - \frac{\omega}{c}\vec{r} \cdot \vec{u}_0\right) d^3\vec{r}_0. \tag{30}$$

$$\tilde{K}_B(\vec{r}, \omega) = \frac{1}{2j\pi|\vec{r}|} \sin\left(\frac{\omega|\vec{r}|}{c}\right) + \underbrace{\frac{\omega^2}{8j\pi^2c^2} \int_{\mathcal{V}} \frac{\gamma_\kappa(\vec{r}_0)}{|\vec{r}_0| \times |\vec{r} - \vec{r}_0|} \sin\left[\frac{\omega}{c}(|\vec{r}_0| + |\vec{r} - \vec{r}_0|)\right] d^3\vec{r}_0}_{\tilde{C}(\vec{r}, \omega)} \tag{29}$$

This last equation can be rewritten in a more simple way if the function $\gamma_\kappa(\vec{r}_0)$ that describes the inhomogeneities has the spherical symmetry. It results immediately from this assumption that the kernel distribution $\tilde{K}_B(\vec{r},\omega)$ and the correction term $\tilde{C}(\vec{r},\omega)$ also have the spherical symmetry. Without loss of generality, $\vec{r}_0$ can be described through the spherical coordinates and the expression given in (30) can be computed with an observation point located on any arbitrary axis, e.g., the z-axis:

$$\vec{r} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \text{ and } \vec{r}_0 = \begin{pmatrix} r_0 \cos\varphi_0 \sin\theta_0 \\ r_0 \sin\varphi_0 \sin\theta_0 \\ r_0 \cos\theta_0 \end{pmatrix}.$$

Starting from these remarks, the expression of the correction term can be reduced to a very simple form with the calculation steps in (31) (shown at the bottom of the page) and (32).

Equation (31) leads finally to a very simple expression of the kernel distribution $\tilde{K}_B(\vec{r},\omega)$:

$$\tilde{K}_B(\vec{r},\omega) = \frac{1}{2j\pi|\vec{r}|} \sin\left(\frac{\omega|\vec{r}|}{c}\right) \cdot \left[1 + \frac{\omega}{c}\int_0^\infty \gamma_\kappa(r_0) \sin\left(\frac{2\omega r_0}{c}\right) dr_0\right]. \quad (32)$$

It results from (31) that the correction term due to inhomogeneities reduces (up to a scaling factor) to the Fourier sine transform of the function $\gamma_\kappa(r_0)$ that describes the radial relative variations of the compressibility of the propagation medium.

In the following discussion, the function $\gamma_\kappa(\vec{r}_0)$ is written as a sum of a constant and another function $\gamma'_\kappa(\vec{r}_0)$ whose mean value evaluated over the volume $\mathcal{R}$ is zero:

$$\gamma_\kappa(\vec{r}_0) = \gamma_0 + \gamma'_\kappa(\vec{r}_0)$$

with

$$\gamma_0 = \int_{\mathcal{R}} \gamma_\kappa(\vec{r}_0) d^3\vec{r}_0 \Big/ \int_{\mathcal{R}} d^3\vec{r}_0 \text{ and } \int_{\mathcal{R}} \gamma'_\kappa(\vec{r}_0)\, d^3\vec{r}_0 = 0.$$

We now consider the characteristic dimension L of the whole region $\mathcal{R}$ that contains the inhomogeneities and the characteristic distance l for which the function $\gamma'_\kappa(\vec{r}_0)$ shows significant variations (this parameter can be understood as the correlation length of the inhomogeneities). The different scales of the problem are illustrated on Fig. 4. The contribution of the constant γ_0 to the correction term $\tilde{C}(\vec{r},\omega)$ results from an integration of the sine function over an interval of length L; it is clear that this contribution is negligible as soon as the condition $kL \ll 1$ (k is the wavenumber defined by

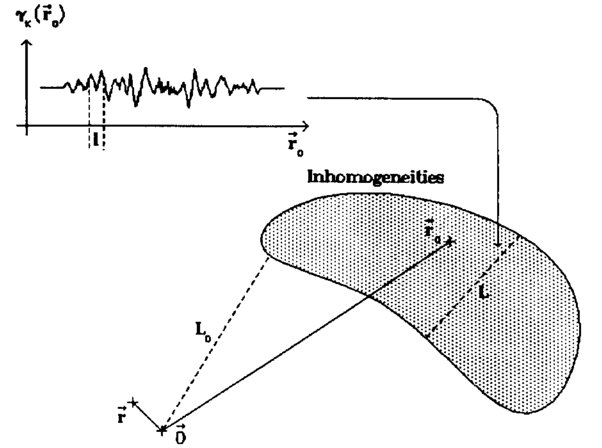

Fig. 4. Geometrical representation of the different scales for justification of the extended FBA. The observation point $\vec{r}$ is near the origin, L_0 is the minimum distance between the origin and the volume $\mathcal{R}$ containing the inhomogeneities, L is the characteristic dimension of $\mathcal{R}$ and l, the characteristic distance for which $\gamma_\kappa(\vec{r}_0)$ shows the significative variations. An example of variations of $\vec{R}_0$ along a section of $\mathcal{R}$ is also represented.

$k = \omega/c$) is satisfied. Since the FBA is a low frequency approximation, the contribution of $\gamma'_\kappa(\vec{r}_0)$ to the correction term is also negligible if it varies much more rapidly than the sine function, therefore resulting to the condition $kl \ll 1$. In fact, previous works give a single condition as $k^2 lL \ll 1$. We obtain here a classical justification of the FBA [13]–[15]. Under these conditions, the correction term is negligible and the time-reversed pressure field reduces to the difference of the two spherical waves previously observed in the case of a homogeneous medium}. Anyway, if the correction term cannot be considered as completely negligible, it remains small due to the Fourier sine transform of a function of space that varies more rapidly than the sine kernel of the transform. This theorem is a generalization of the results obtained with monochromatic phase conjugation mirrors in optics, which compensate the wavefront distortions due to weak scatterers [17], [18]

In all the preceding discussion, we have written some validity conditions for the different approximations we consider. As it is clearly visible, these validity conditions are frequency-dependant. As mentioned at the head of this section, it is possible to extend these validity conditions to the time domain as soon as they are satisfied in the whole frequency range of the excitation function $\phi(t)$.

In the frame of the time-reversal process, we take into account the pressure field that propagates during the first step

$$\begin{aligned} \tilde{C}(\vec{r},\omega) &= \frac{\omega^2}{8j\pi^2c^2}\int_0^\infty r_0^2\, dr_0 \int_0^\pi \sin\theta_0 d\theta_0 \int_0^{2\pi} d\varphi_0 \frac{\gamma_\kappa(r_0)}{r_0^2} \sin\left(\frac{2\omega r_0 - \omega|\vec{r}|\cos\theta_0}{c}\right) \\ &= \frac{\omega^2}{4j\pi c^2}\int_0^\infty \gamma_\kappa(r_0)\, dr_0 \int_0^\pi \sin\left(\frac{2\omega r_0 - \omega|\vec{r}|\cos\theta_0}{c}\right) \sin\theta_0\, d\theta_0 \\ &= \frac{\omega}{2j\pi|\vec{r}|c} \sin\left(\frac{\omega|\vec{r}|}{c}\right) \int_0^\infty \gamma_\kappa(r_0) \sin\left(\frac{2\omega r_0}{c}\right) dr_0. \end{aligned} \quad (31)$$

to determine the secondary sources (monopole and dipole sources) created on the surface of the cavity during the second step. In particular, we take into account the distortions introduced by the interaction of the incident acoustic wave with the inhomogeneities. An interesting analysis consists of computing the pressure field generated in the inside of the cavity from a standard focusing technique, for example in transmitting from the cavity boundaries the ideal waveforms which would converge on the target in the case of a homogeneous medium (this technique considers only a time-delay law to focuse on the target). This is equivalent to a spherical focusing through the aberrating layer, taking only into account the spatial localization of the target (ignoring in such case that the propagation medium contains inhomogeneities). It is shown in Appendix II that the reconstructed pressure field can be written as in (26) with another kernel distribution $K'_B(\vec{r},t)$ given by (33), shown at the bottom of the page, with $\mathcal{D}'(\vec{r},\vec{r}_0) = |\vec{r}_0| - |\vec{r}-\vec{r}_0|$. Taking the Fourier transform of (33), we obtain a similar expression as in (29):

$$\tilde{K}'_B(\vec{r},\omega) = \frac{1}{2j\pi|\vec{r}|}\sin\left(\frac{\omega|\vec{r}|}{c}\right) + \tilde{C}'(\vec{r},\omega) \qquad (34)$$

with another expression of the correction term given by

$$\tilde{C}'(\vec{r},\omega) = \frac{\omega^2}{8j\pi^2c^2}\int_{\mathcal{V}} \frac{\gamma_\kappa(\vec{r}_0)}{|\vec{r}_0| \times |\vec{r}-\vec{r}_0|} \cdot \sin\left(\frac{\omega|\vec{r}_0|}{c}\right)\exp\left(j\frac{\omega|\vec{r}-\vec{r}_0|}{c}\right) d^3\vec{r}_0. \qquad (35)$$

If we consider now the same assumptions as above (observation point near the origin and inhomogeneities far from the origin, inhomogeneities that have the spherical symmetry), it can be shown, after some elementary computations that we do not present here in detail, that the correction term reduces to a similar expression as (31):

$$\tilde{C}'(\vec{r},\omega) = \frac{\omega}{4\pi|\vec{r}|c}\sin\left(\frac{\omega|\vec{r}|}{c}\right) \cdot \int_0^\infty \gamma_\kappa(r_0)\left[1 - \exp\left(\frac{2j\omega r_0}{c}\right)\right] dr_0. \qquad (36)$$

It results from this equation that the correction term can be now written (up to a scaling factor) as the sum of three different components.

1) A first component that can be described as the Fourier sine transform of the function $\gamma_\kappa(r_0)$, it is similar to the obtained expression in the frame of the time-reversal process; considering the same arguments as previously, this first component is negligible.
2) A second component that can be described as the Fourier cosine transform of the function $\gamma_\kappa(r_0)$; since the cosine and sine functions show similar behavior, it results immediately that this second term is also negligible.
3) A third component that exactly corresponds to the mean value of $\gamma_\kappa(r_0)$; it results therefore that this term is generally small (one basic assumption of the FBA), but is not necessarily negligible.

The preceding remarks lead to a very interesting and important conclusion. If we consider the time-reversal process, the reconstructed pressure field results from the superposition of the two spherical waves obtained for a homogeneous medium and a correction term that is generally negligible in the frame of the FBA's validity conditions. If we now consider the more classical focusing technique described above, the reconstructed pressure field shows similar terms (the two spherical waves and correction terms that are negligible for the same reasons), but also another term (the third component previously mentioned) that is small, but not necessarily negligible as the other components of the correction term. It results therefore that the correction term generates greater distortions to the optimal focusing in the case of the classical focusing technique than in the case of the time-reversal process. These results illustrate the focusing quality improvement due to the time-reversal process.

VI. Conclusion

In this paper, we have analyzed the theoretical concept of closed time-reversal cavity to optimize focusing in an inhomogeneous medium. One important result is that we cannot only generate the time-reversal of an acoustic wavefield; this time-reversal is always followed by a back-propagation of the initial field, thus resulting in a limitation of the self-focusing process. The comparison of the time-reversal process with the time-delay law technique in the case of a weakly inhomogeneous medium illustrates the focusing improvement due to our method. The problem has been treated in the transient regime; it includes and generalizes the monochromatic formalism used in Optics. Finally, it is important to note that this model is a theoretical approach, and that such a time-reversal cavity cannot be realized experimentally. The model allows us to understand the basic physics contained in the principle, and exhibits the limitations essentially due to diffraction. We are now working on more realistic assumptions such as transducer arrays that will be compared with the results presented in this paper.

Appendix I
Computation of $p_{tr}(\vec{r},t)$

In this appendix, we develop the different computation steps

$$K'_B(\vec{r},t) = \frac{1}{4\pi|\vec{r}|}\delta\left(t + \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi|\vec{r}|}\delta\left(t - \frac{|\vec{r}|}{c}\right) - \frac{1}{16\pi^2c^2} \times \partial_{tt}\int_{\mathcal{V}} \frac{\gamma_\kappa(\vec{r}_0)}{|\vec{r}_0| \times |\vec{r}-\vec{r}_0|}\left[\delta\left(t + \frac{\mathcal{D}'(\vec{r},\vec{r}_0)}{c}\right) - \delta\left(t - \frac{\mathcal{D}(\vec{r},\vec{r}_0)}{c}\right)\right] d^3\vec{r}_0, \qquad (33)$$

to obtain the expression of the time-reversed pressure field $p_{tr}(\vec{r},t)$ given by (12).

In order to provide another expression of (9), we are first interested in $\nabla_0^2\sigma_1(\vec{r}_0,t)$, in (37), shown at the bottom of the page.

The two terms appearing in (37) can be replaced using the relations given in (10), such that $\nabla_0^2\sigma_1(\vec{r}_0,t)$ can be rewritten as (38), shown at the bottom of the page.

Similarly, it results immediately from (3) the following relation:

$$\nabla_0^2 G_d(\vec{r},\vec{r}_0,t) = c^{-2}\partial_{tt}G_d(\vec{r},\vec{r}_0,t) - \delta(t)\delta(\vec{r}-\vec{r}_0).$$

Introducing (38) in (9) and defining $\Omega(\vec{r},\vec{r}_0,t)$ by

$$\begin{aligned}\Omega(\vec{r},\vec{r}_0,t) =& G_d(\vec{r},\vec{r}_0,t)\underset{t}{\overset{*}{}}\nabla_0^2\sigma_1(\vec{r}_0,t)\\ &- \sigma_1(\vec{r}_0,t)\underset{t}{\overset{*}{}}\nabla_0^2 G_d(\vec{r},\vec{r}_0,t)\end{aligned}$$

we obtain (39), shown at the bottom of the page, after reordering the different terms.

Using the basic properties of the time-convolution with respect to the time-derivative operator, we can easily verify that the first two terms of (39) compensate. The volume integrals of Dirac distributions can be computed immediately, such that the time-reversed pressure field $p_{tr}(\vec{r},t)$ given by (9) becomes (40).

It is important to note that the formal operator $\mathcal{A}(\vec{r}_0)$ generally acts on the time variable t (this property is also valid for the time-convolution operator); as an essential consequence, the time-window distribution $W(t)$ appearing in this expression cannot be placed anywhere in (40) and we cannot derive a more simple expression in the general case, without taking into account some specific properties of the operator $\mathcal{A}(\vec{r}_0)$.

The preceding expression of the time-reversed pressure field contains the term $\sigma_1(\vec{r},t)$ that can be written as $p(\vec{r},T-t)W(T-t)$ according to (6). If we consider three functions of time $a(t)$, $b(t)$ and $c(t)$ such that we have the relation $c(t) = a(t)\underset{t}{\overset{*}{}}b(t)$, it results from the classical properties of the convolution operator that we have also [8]:

$$c(T-t) = a(T-t)\underset{t}{\overset{*}{}}b(-t) = a(-t)\underset{t}{\overset{*}{}}b(T-t).$$

Using this relation and the expression of $p(\vec{r},t)$ given by (5),

$$\begin{aligned}\nabla_0^2\sigma_1(\vec{r}_0,t) =& \nabla_0^2[p(\vec{r}_0,T-t)W(T-t)] = W(T-t)\nabla_0^2 p(\vec{r}_0,T-t)\\ =& W(T-t)(\nabla_0^2 - c^{-2}\partial_{tt})p(\vec{r}_0,T-t) + c^{-2}W(T-t)\partial_{tt}p(\vec{r}_0,T-t).\end{aligned} \tag{37}$$

$$\begin{aligned}\nabla_0^2\sigma_1(\vec{r}_0,t) =& c^{-2}\partial_{tt}\sigma_1(\vec{r}_0,t) + c^{-2}\eta(\vec{r}_0)\delta(t) - c^{-2}\varepsilon(\vec{r}_0)\delta'(t)\\ &-W(T-t)\phi(T-t)\delta(\vec{r}_0) - W(T-t)\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0,T-t)\}.\end{aligned} \tag{38}$$

$$\begin{aligned}\Omega(\vec{r},\vec{r}_0,t) =& G_d(\vec{r},\vec{r}_0,t)\underset{t}{\overset{*}{}}c^{-2}\partial_{tt}\sigma_1(\vec{r}_0,t) - \sigma_1(\vec{r}_0,t)\underset{t}{\overset{*}{}}c^{-2}\partial_{tt}G_d(\vec{r},\vec{r}_0,t)\\ &- \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\phi\left(T-t+\frac{|\vec{r}-\vec{r}_0|}{c}\right)W\left(T-t+\frac{|\vec{r}-\vec{r}_0|}{c}\right)\delta(\vec{r}_0)\\ &+\sigma_1(\vec{r}_0,t)\delta(\vec{r}-\vec{r}_0) - [W(T-t)\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0,T-t)\}]\underset{t}{\overset{*}{}}G_d(\vec{r},\vec{r}_0,t)\\ &+\frac{1}{c^2}\left[\eta(\vec{r}_0)\delta\left(t-\frac{|\vec{r}-\vec{r}_0|}{c}\right) - \varepsilon(\vec{r}_0)\delta'\left(t-\frac{|\vec{r}-\vec{r}_0|}{c}\right)\right]\frac{1}{4\pi|\vec{r}-\vec{r}_0|}.\end{aligned} \tag{39}$$

$$\begin{aligned}p_{tr}(\vec{r},t) =& \sigma_1(\vec{r},t) - \frac{1}{4\pi|\vec{r}|}\phi\left(T-t+\frac{|\vec{r}|}{c}\right)W\left(T-t+\frac{|\vec{r}|}{c}\right)\\ &- \int_{\mathcal{V}}[W(T-t)\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0,T-t)\}]\underset{t}{\overset{*}{}}G_d(\vec{r},\vec{r}_0,t)\;d^3\vec{r}_0\\ &+ \int_{\mathcal{V}}[\mathcal{A}(\vec{r}_0)\{p_{tr}(\vec{r}_0,t)\}]\underset{t}{\overset{*}{}}G_d(\vec{r},\vec{r}_0,t)\;d^3\vec{r}_0\\ &+\frac{1}{c^2}\int_{\mathcal{V}}\left[\eta(\vec{r}_0)\delta\left(t-\frac{|\vec{r}-\vec{r}_0|}{c}\right) - \varepsilon(\vec{r}_0)\delta'\left(t-\frac{|\vec{r}-\vec{r}_0|}{c}\right)\right]\frac{d^3\vec{r}_0}{4\pi|\vec{r}-\vec{r}_0|}.\end{aligned} \tag{40}$$

we can write the following expression of $\sigma_1(\vec{r},t)$:

$$\sigma_1(\vec{r},t) = \frac{1}{4\pi|\vec{r}|}\phi\left(T - t - \frac{|\vec{r}|}{c}\right)W(T-t) + W(T-t)\int_{\mathcal{V}}\left\{[\mathcal{A}(\vec{r}_0)\{p(\vec{r}_0, T-t)\}] \cdot \overset{*}{_t} G_c(\vec{r},\vec{r}_0,t)d^3\vec{r}_0\right\}. \tag{41}$$

Introducing (41) in (40), we finally obtain the expression of the time-reversed pressure field given by (12).

Appendix II
Computation of $K'_B(\vec{r},t)$

Here we are interested in a spherical focusing through the inhomogeneous medium using a classical focusing technique that takes only into account the spatial localization of the target on which the pressure field has to be focused. As a consequence, the secondary sources created on the surface of the cavity are those determined in the second section; they correspond to a propagation in a homogeneous medium during the first step. If the recording duration T is chosen greater than $(T_\phi + d_m/c)$, where the notations are those introduced in the first section, $\sigma_1(\vec{r},t)$ can be written as

$$\sigma_1(\vec{r},t) = \frac{1}{4\pi|\vec{r}|}\phi\left(T - t - \frac{|\vec{r}|}{c}\right) \tag{42}$$

and $\sigma_0(\vec{r},t) = \vec{n}\cdot\vec{\nabla}\sigma_1(\vec{r},t)$. Under such condition, (38) is changed to

$$\nabla_0^2\sigma_1(\vec{r}_0,t) = c^{-2}\partial_{tt}\sigma_1(\vec{r}_0,t) - \phi(T-t)\delta(\vec{r}_0). \tag{43}$$

If $\Omega(\vec{r},\vec{r}_0,t)$ is defined as in the previous appendix, we similarly obtain (44), shown at the bottom of the page.

During the second step, the time-reversed pressure field propagates in the inhomogeneous medium. Introducing (44) in (9), we immediately obtain the resulting pressure field

$$p'_{tr}(\vec{r},t) = \frac{1}{4\pi|\vec{r}|}\phi\left(T - t - \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi|\vec{r}|}\phi\left(T - t + \frac{|\vec{r}|}{c}\right) + \int_{\mathcal{V}}[\mathcal{A}(\vec{r}_0)\{p'_{tr}(\vec{r}_0,t)\}]\overset{*}{_t}G_d(\vec{r},\vec{r}_0,t)\ d^3\vec{r}_0 \tag{45}$$

and the corresponding kernel distribution $K'(\vec{r},t)$

$$K'(\vec{r},t) = \frac{1}{4\pi|\vec{r}|}\delta\left(t + \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi|\vec{r}|}\delta\left(t - \frac{|\vec{r}|}{c}\right) + \int_{\mathcal{V}}[\mathcal{A}(\vec{r}_0)\{K'(\vec{r}_0,t)\}]\overset{*}{_t}G_d(\vec{r},\vec{r}_0,t)\ d^3\vec{r}_0. \tag{46}$$

Considering the particular case of inhomogeneities of compressibility only, the kernel distribution given by (46) can be rewritten as

$$K'(\vec{r},t) = \frac{1}{4\pi|\vec{r}|}\delta\left(t + \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi|\vec{r}|}\delta\left(t - \frac{|\vec{r}|}{c}\right) - \frac{1}{4\pi c^2}\times \partial_{tt}\int_{\mathcal{V}}\frac{\gamma_\kappa(\vec{r}_0)}{|\vec{r}-\vec{r}_0|}K'\left(\vec{r}_0, t - \frac{|\vec{r}-\vec{r}_0|}{c}\right)d^3\vec{r}_0. \tag{47}$$

Looking at this expression, if the propagation medium is weakly inhomogeneous, the kernel distribution $K'(\vec{r},t)$ can be interpreted as a zero-order term (the difference of the two impulse spherical waves) and a small correction (and self-consistent) term. Using the extended FBA, the expression of the kernel distribution can be written in a more simple way, replacing K' in the integral by the zero-order term. It results therefore that the extended FBA can be described by the following substitution in the integral over $\mathcal{V}$:

$$K'\left(\vec{r}_0, t - \frac{|\vec{r}-\vec{r}_0|}{c}\right) \Longrightarrow \frac{1}{4\pi|\vec{r}_0|}\delta\left(\tau + \frac{|\vec{r}_0|}{c}\right) - \frac{1}{4\pi|\vec{r}_0|}\delta\left(\tau - \frac{|\vec{r}_0|}{c}\right)$$

where τ is defined by $\tau = t - (|\vec{r}-\vec{r}_0|)/c$. This substitution leads immediately to (33).

Acknowledgment

The authors would like to thank Mrs. Claire Prada and Mr. Vincent Gibiat for their helpful comments and discussions about this work.

References

[1] P. M. Morse and K. U. Ingard, *Theoretical Acoustics.* New York: McGraw-Hill 1968.
[2] M. Fink, C. Prada, F. Wu, and D. Cassereau, "Self-focusing in inhomogeneous media with time-reversal acoustic mirrors," *Proc. IEEE Ultrason. Symp.*, 1989, pp. 681–686.
[3] F. Wu, C. Prada, and M. Fink, "Experimental progress of ultrasonic time-reversal mirrors," *Acoust. Imaging*, 1991.
[4] C. Prada, F. Wu, and M. Fink, "The iterative time-reversal mirror: A solution to self-focusing in the pulse echo mode," *J. Acoust. Soc. Am.* July 1991.
[5] D. Cassereau, F. Wu, and M. Fink, "Limits of self-focusing using closed time-reversal cavities and mirrors - Theory and experiment," in *Proc. IEEE Ultrason. Symp.*, pp. 1613–1618, 1990.
[6] D. Cassereau and M. Fink, "Theoretical modelisation of time-reversal cavities, application to self-focusing in inhomogeneous media," *Acoust. Imaging*, 1991.
[7] M. Born and E. Wolf, *Principles of Optics.* New York: Pergamon, 1975.
[8] A. Papoulis, *Signal Analysis.* New York: McGraw-Hill 1984.
[9] R. P. Porter and A. J. Devaney, "Holography and the inverse source problem", *J. Opt. Soc. Amer.*, vol. 72, pp. 327–330, 1982.
[10] R. P. Porter, "Image formation with arbitrary holographic type surfaces," *Phys. Lett.*, vol. 29A, pp. 193–194, 1969.

$$\Omega(\vec{r},\vec{r}_0,t) = G_d(\vec{r},\vec{r}_0,t)\overset{*}{_t}c^{-2}\partial_{tt}\sigma_1(\vec{r}_0,t) - \sigma_1(\vec{r}_0,t)\overset{*}{_t}c^{-2}\partial_{tt}G_d(\vec{r},\vec{r}_0,t) - \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\phi\left(T - t + \frac{|\vec{r}-\vec{r}_0|}{c}\right)\delta(\vec{r}_0) + \sigma_1(\vec{r}_0,t)\delta(\vec{r}-\vec{r}_0). \tag{44}$$

[11] R. P. Porter, "Diffracted-limited, scalar image formation with holograms of arbitrary shape," *J. Opt. Soc. Amer.*, vol. 60, pp. 1051–1059, 1970.

[12] R. P. Porter and A. J. Devaney, "Generalized holography and computational solutions to inverse source problems," *J. Opt. Soc. Amer.*, vol. 72, pp. 1707–1713, 1982.

[13] L. A. Chernov, *Wave Propagation in a Random Medium.* New York: McGraw Hill, 1960.

[14] D. Mintzer, "Wave propagation in a randomly inhomogeneous medium—Part I," *J. Acoust. Soc. Amer.*, vol. 25, pp. 922–927, 1953.

[15] ——, "Wave propagation in a randomly inhomogeneous medium—Part II," *J. Acoust. Soc. Amer.*, vol. 25, pp. 1107–1111, 1953.

[16] J. H. Rose and M. Cheney, "Self-consistent equations for variable-velocity three-dimensional inverse scattering," *Phys. Rev. Lett.*, vol. 59, pp. 954–957, 1987.

[17] G. S. Agarwal and E. Wolf, "Theory of phase conjugation with weak scatterers," *J. Opt. Soc. Amer.*, vol. 72, pp. 321–326, 1982.

[18] G. S. Agarwal, A. T. Friberg, and E. Wolf, "Effect of backscattering in phase conjugation with weak scatterers," *J. Opt. Soc. Am.*, vol. 72, pp. 861–863, 1982.

[19] F. Wu, M. Fink, R. Mallart, J. L. Thomas, N. Chakroun, D. Cassereau, and C. Prada, "Optimal focusing through aberrating media: A comparison between time-reversal mirror and time delay correction techniques," *Proc. IEEE Ultrason. Symp.*, 1991.

[20] F. Wu, J. L. Thomas, and M. Fink, "Time-reversal of ultrasonic fields—Part II: Basic principles and experiments with time-reversal mirrors," *IEEE Trans. Ultrason., Ferroelec., Freq. Contr.*, this issue, pp. 567-578.

Didier Cassereau was born in 1963 in Cholet, France. He received the D.E.A. degree in nuclear physics and elementary particles in 1984 from Paris University, Paris, France. In 1988, he received the Doctorat degree in physics from Paris VII University for his work on impulse diffraction of acoustical waves in fluid and solid media.

Since 1988, he is teaching assistant for computer science and numerical analysis at ESPCI in Paris. He also works in the frame of the Laboratoire Ondes et Acoustique on very general subjects like transient propagation and diffraction in homogeneous and inhomogeneous media, diffraction patterns of ultrasonic 1-D and 2-D arrays, optimal focusing in fluids and/or solids, and the theoretical approach of the time-reversal principle.

Mathias A. Fink, for a photograph and biography, please see page 566 of this TRANSACTIONS.