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Mixed Scattering Matrix: Properties and Applications

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Neuchâtel, 2005

Outline

Introduction

1. Admittance, Wave Scattering, and Mixed Scattering Matrices of the Multi-Port Network

2. Mixed Scattering Matrix of a SAW Transducer

3. SAW Filter Simulation

4. Modeling in the Quasi-Static Approximation

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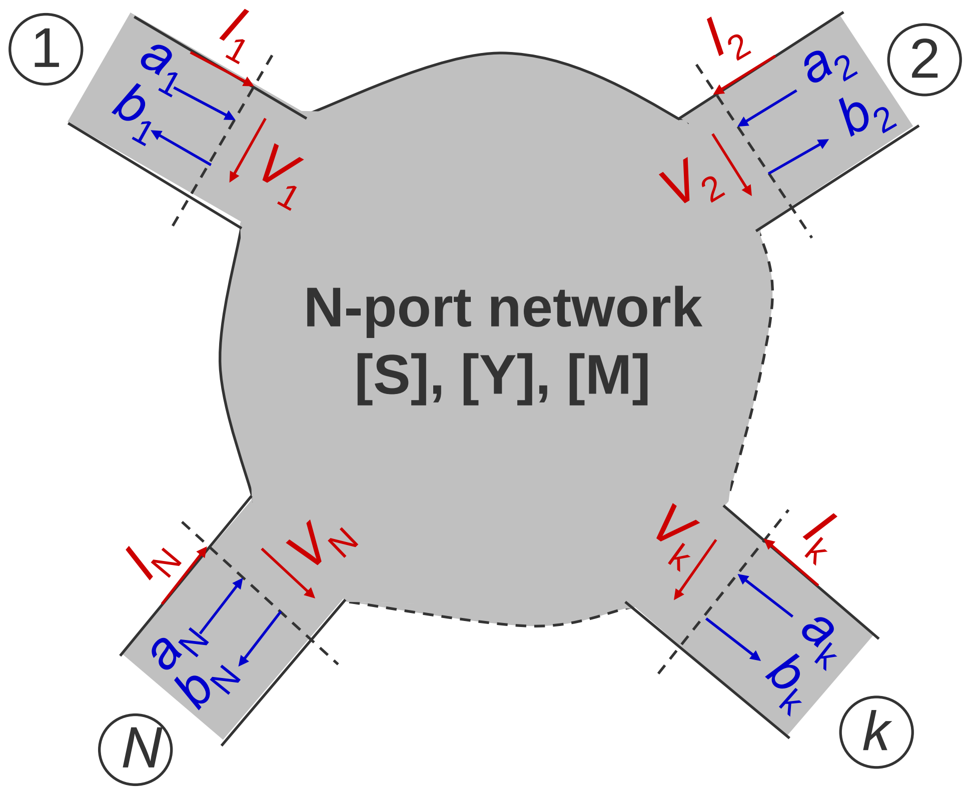
Part 1. Admittance, Wave Scattering, and Mixed Scattering Matrices of the Multi-Port Network

Multi-Port Network



B=SA (scattering matrix)

I =YV (admittance matrix)

Fig. 1. An arbitrary N-port microwave network specified in terms of the **S**- and **Y**-matrices

Wave Scattering and Admittance Matrices

Scattering Matrix

$$\mathbf{B}=\mathbf{S}\mathbf{A} \qquad (1)$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1N} \\ s_{21} & s_{22} & \cdots & s_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N1} & s_{N2} & \cdots & s_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \qquad (2)$$

Admittance Matrix

$$\mathbf{I}=\mathbf{Y}\mathbf{V} \qquad (3)$$

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1N} \\ y_{21} & y_{22} & \cdots & y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ y_{N1} & y_{N2} & \cdots & y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \qquad (4)$$

$\mathbf{A}=[a_1\ a_2 \ldots a_N]^T$ vector of the incident waves

$\mathbf{B}=[b_1\ b_2 \ldots b_N]^T$ vector of the reflected waves

$\mathbf{I}=[I_1\ I_2 \ldots I_N]^T$ vector of the terminal currents

$\mathbf{V}=[V_1\ V_2 \ldots V_N]^T$ vector of the terminal voltages

$\mathbf{S}=[s_{ik}]$ scattering matrix (dimensionless)

$\mathbf{Y}=[Y_{ik}]$ admittance matrix (Ω^{-1})

Mixed Scattering Matrix

Mixed scattering matrix **M** is a mixed units hybrid of the scattering matrix **S** and admittance matrix **Y**.

$$
\begin{bmatrix} \mathbf{B} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{ae}} \\ \mathbf{M_{ea}} & \mathbf{M_{ee}} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{V} \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{ae}} \\ \mathbf{M_{ea}} & \mathbf{M_{ee}} \end{bmatrix} \tag{5}
$$

$$
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ \hdashline I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \left[\begin{array}{cccc:cccc}
m_{11} & m_{12} & \cdots & m_{1m} & m_{1,m+1} & m_{1,m+2} & \cdots & m_{1,m+n} \\
m_{21} & m_{22} & \cdots & m_{2m} & m_{2,m+1} & m_{2,m+2} & \cdots & m_{2,m+n} \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
m_{m1} & m_{m2} & \cdots & m_{mm} & m_{m+1,m+1} & m_{m,m+2} & & m_{m,m+n} \\
\hdashline
m_{m+1,1} & m_{m+1,2} & \cdots & m_{m+1,m} & m_{m+1,m+1} & m_{m+1,m+2} & \cdots & m_{m+1,m+n} \\
m_{m+2,1} & m_{m+2,2} & \cdots & m_{m+2,m} & & m_{m+2,m+2} & \cdots & m_{m+2,m+n} \\
\vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\
m_{m+n,1} & m_{m+n,2} & \cdots & m_{m+n,m} & & m_{m+n,m+2} & & m_{m+n,m+n}
\end{array}\right] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ \hdashline V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \tag{6}
$$

Nomenclature

m	number of the acoustic wave ports
n	number of the electric ports
$N=m+n$	total number of the ports
$\mathbf{A}=[a_1\, a_2 \ldots a_m]$	vector of the incident waves on the acoustic ports
$\mathbf{B}=[b_1\, b_2 \ldots b_m]$	vector of the reflected waves from the acoustic ports
$\mathbf{I}=[I_1\; I_2 \ldots I_n]$	vector of the terminal currents on the electric ports
$\mathbf{V}=[V_1\, V_2 \ldots V_n]$	vector of the terminal voltages on the electric ports
$\mathbf{M}$	mixed scattering matrix of size N x N
$\mathbf{M_{aa}}$	acoustic matrix block of size m x m (dimensionless)
$\mathbf{M_{ae}}$	acoustoelectric matrix block of size m x n (mixed units)
$\mathbf{M_{ea}}$	electroacoustic matrix block of size n x m (mixed units)
$\mathbf{M_{ee}}$	electric matrix block of size n x n (Ω^{-1})

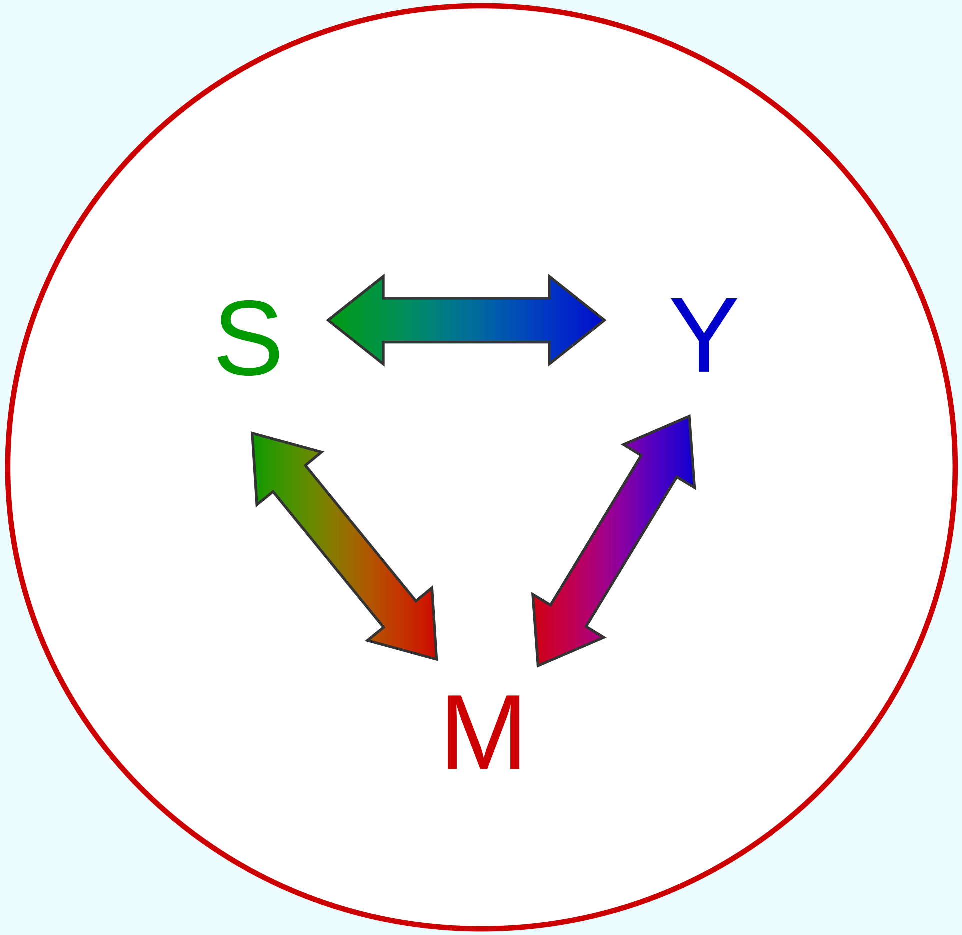
Generalized Wave Amplitudes and Electric Variables

Variables	Matrix	Scalar
Incident and reflected waves	$\begin{cases} \mathbf{A} = \frac{1}{2}(\sqrt{\mathbf{Y_0}}\mathbf{V} + \sqrt{\mathbf{Z_0}}\mathbf{I}) \\ \mathbf{B} = \frac{1}{2}(\sqrt{\mathbf{Y_0}}\mathbf{V} - \sqrt{\mathbf{Z_0}}\mathbf{I}) \end{cases}$ (7)	$\begin{cases} a_k = \frac{1}{2}(\sqrt{Y_{ok}}V_k + \sqrt{Z_{ok}}I_k) \\ b_k = \frac{1}{2}(\sqrt{Y_{ok}}V_k - \sqrt{Z_{ok}}I_k) \end{cases}$ (8)
Terminal current and voltages	$\begin{cases} \mathbf{I} = \sqrt{\mathbf{Y_0}}(\mathbf{A} - \mathbf{B}) \\ \mathbf{V} = \sqrt{\mathbf{Z_0}}(\mathbf{A} + \mathbf{B}) \end{cases}$ (9)	$\begin{cases} I_k = \sqrt{Y_{ok}}(a_k - b_k) \\ V_k = \sqrt{Z_{ok}}(a_k + b_k) \end{cases}$ (10)
Characteristic admittance	$\mathbf{Y}_0 = \mathbf{Z}_0^{-1} = \begin{bmatrix} Y_{01} & 0 & \cdots & 0 \\ 0 & Y_{02} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & Y_{0N} \end{bmatrix}$ (11)	$Y_{0k} = 1/Z_{0k}$ (12)
Average delivered power	$\mathbf{P} = \frac{1}{2}Re\{\mathbf{V}\mathbf{I}^*\} = \frac{1}{2}Re\{(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})^*\}$ $\mathbf{P} = \frac{1}{2}\mathbf{A}\mathbf{A}^* - \frac{1}{2}\mathbf{B}\mathbf{B}^*$ (13) **BA***-**AB***=2Im{**BA***} purely imaginary.	$P_k = \frac{1}{2}Re\{V_k I_k^*\} = \frac{1}{2}Re\{(a_k + b_k)(a_k^* - b_k^*)\}$ $P_k = \frac{1}{2}\lvert a_k \rvert^2 - \frac{1}{2}\lvert b_k \rvert^2$ (14) $b_k\, a_k^* - a_k\, b_k^* = 2\mathrm{Im}\{ b_k\, a_k^*\}$ purely imaginary.

Conversion Between Admittance, Scattering, and Mixed Scattering Matrices

	S	Y	M
S_{aa} S_{ae} S_{ea} S_{ee}	S_{aa} S_{ae} S_{ea} S_{ee}	$S=(E-\overline{Y})(E+\overline{Y})^{-1}=$ $=(E+\overline{Y})^{-1}(E-\overline{Y})$ $\overline{Y}=\sqrt{Z_0}Y\sqrt{Z_0}$ (15)	$M_{aa}-M_{ae}(Y_0^e+M_{ee})^{-1}M_{ea}$ $2M_{ae}(Y_0^e+M_{ee})^{-1}\sqrt{Y_0^e}$ $-\sqrt{Y_0^e}(Y_0^e+M_{ee})^{-1}M_{ea}$ $\sqrt{Y_0^e}(Y_0^e+M_{ee})^{-1}(Y_0^e-M_{ee})\sqrt{Z_0^e}$ (16)
Y_{aa} Y_{ae} Y_{ea} Y_{ee}	$\overline{Y}=(E-S)(E+S)^{-1}=$ $=(E+S)^{-1}(E-S)$ $\overline{Y}=\sqrt{Z_0}Y\sqrt{Z_0}$	Y_{aa} Y_{ae} Y_{ea} Y_{ee} (17)	$\sqrt{Y_0^a}(E+M_{aa})^{-1}(E-M_{aa})\sqrt{Y_0^a}$ $-2\sqrt{Y_0^a}(E+M_{aa})^{-1}M_{ae}$ $M_{ea}(E+M_{aa})^{-1}\sqrt{Y_0^a}$ $M_{ee}-M_{ea}(E+M_{aa})^{-1}M_{ae}$ (18)
M_{aa} M_{ae} M_{ea} M_{ee}	$S_{aa}-S_{ae}(E+S_{ee})^{-1}S_{ea}$ $S_{ae}(E+S_{ee})^{-1}\sqrt{Y_0^e}$ $-2\sqrt{Y_0^e}(E+S_{ee})^{-1}S_{ea}$ $\sqrt{Y_0^e}(E+S_{ee})^{-1}(E-S_{ee})\sqrt{Y_0^e}$ (19)	$(E+\overline{Y}aa)^{-1}(E-\overline{Y}aa)$ $-(E+\overline{Y}aa)^{-1}\sqrt{Z_0^a}Y_{ae}$ $2Y_{ea}\sqrt{Z_0^a}(E+\overline{Y}aa)^{-1}$ $Y_{ee}-Y_{ea}\sqrt{Z_0^a}(E+\overline{Y}aa)^{-1}\sqrt{Z_0^a}Y_{ae}$ $\overline{Y}aa=\sqrt{Z_0^a}Y_{aa}\sqrt{Z_0^a}$ (20)	M_{aa} M_{ae} M_{ea} M_{ee}

Mutual Conversion



Reciprocity and Power Conservation

Network	Parameters		
	Y	S	M
Reciprocal	$\mathbf{Y}=\mathbf{Y}^{T}$ $Y_{ik}=Y_{ki}$	$\mathbf{S}=\mathbf{S}^{T}$ $S_{ik}=S_{ki}$	$\mathbf{M_{aa}}=\mathbf{M}^{T}_{\mathbf{aa}}$ $\mathbf{M_{ea}}=-2\mathbf{M}^{T}_{\mathbf{ae}}$ $\mathbf{M_{ee}}=\mathbf{M}^{T}_{\mathbf{ee}}$ (21)
Lossless	$\mathbf{Re\{Y\}=0}$ $Re\{Y_{ik}\}=0$	$\mathbf{S^{*}S=E}$ $\sum_{j=1}^{N} S^{*}_{ji}S_{jk}=\delta_{ik}$	$\mathbf{M^{*}_{aa}M_{aa}=E}$ $\mathbf{M^{*}_{ea}=2M^{*}_{aa}M_{ae}}$ $\mathbf{Re\{M_{ee}\}=M^{*}_{ae}M_{ae}}$ (22)
Reciprocal & Lossless	$\mathbf{Y=-Y^{*}}$ $Y_{ik}=-Y_{ki}{}^{*}$	$\mathbf{S^{*}S=SS^{*}=E}$ $\sum_{j=1}^{N} S^{*}_{ji}S_{jk}=\sum_{j=1}^{N} S^{*}_{ij}S_{kj}=\delta_{ik}$	$\mathbf{M^{*}_{aa}M_{aa}=M_{aa}M^{*}_{aa}=E}$ $\mathbf{M^{*}_{ea}=2M^{*}_{aa}M_{ae}}$ $\mathbf{M_{ae}=\frac{1}{2}M_{aa}M^{*}_{ea}}$ $\mathbf{Re\{M_{ee}\}=M^{*}_{ae}M_{ae}}$ (23)

^{T} matrix transposition, * Hermitian conjugation

$$\delta_{ik}=\begin{cases}1 & i=k\\ 0 & i\neq k\end{cases}$$

Matrix Properties (Reciprocal and Lossless Networks)

1. The admittance matrix **Y** is symmetric for reciprocal networks and purely imaginary for lossless networks.

2. The scattering matrix **S** is symmetric for reciprocal networks and unitary for lossless networks.

3. The dot product of any column/row of the scattering matrix **S** with a conjugate of a different column/row gives zero (orthogonality condition) for reciprocal and lossless networks.

4. The acoustic block $\mathbf{M}_{aa}$ of the mixed scattering matrix **M** satisfies the unitary matrix property, with the acoustoelectric and electroacoustic blocks $\mathbf{M}_{ae}$ and $\mathbf{M}_{ea}$ converted to each other.

Reference Plane Transformation

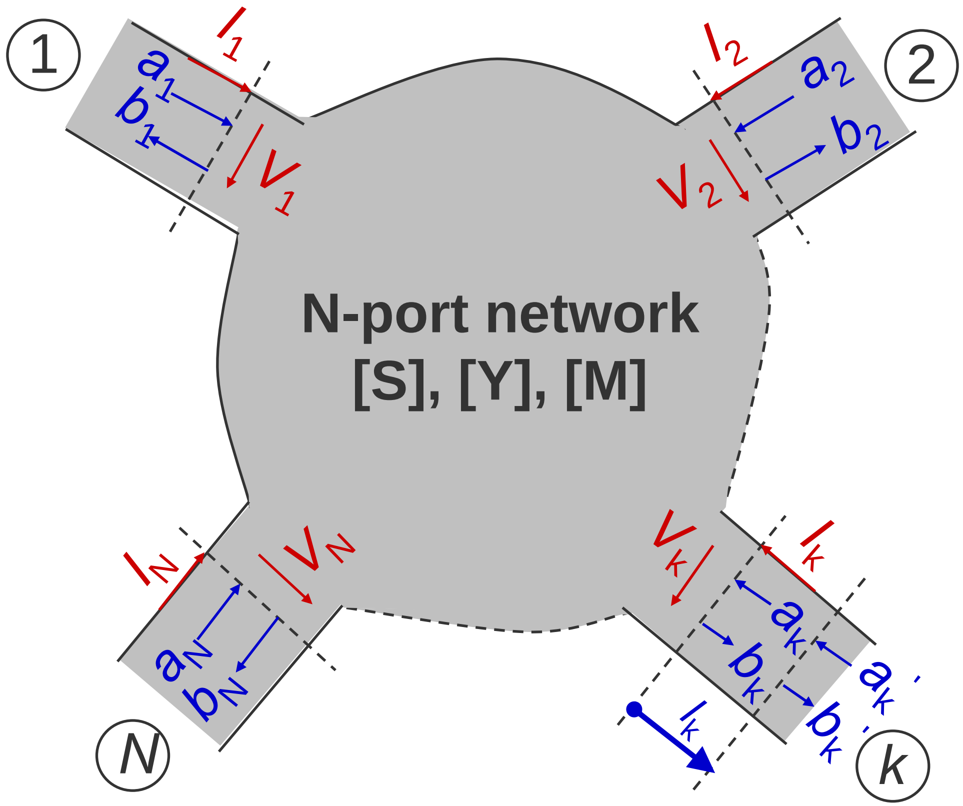


Fig. 2. Shifting reference planes for an N-port network (outward)

Reference Plane Transformation Equations

Type	Matrix	Scalar
Transformation	$\begin{cases} \mathbf{A} = \mathbf{E}\mathbf{A}' \\ \mathbf{B} = \mathbf{E}^{-1}\mathbf{B}' \end{cases}$ $\mathbf{E} = \begin{bmatrix} e^{-j\beta l_1} & 0 & \dots & 0 \\ 0 & e^{-j\beta l_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{-j\beta l_N} \end{bmatrix}$ (24)	$\begin{cases} a_k = e_k a_k' \\ b_k = e_k^{-1} b_k' \end{cases}$ $e_k = e^{-j\beta l_k}$ (25)
Wave scattering matrix	$\mathbf{S}' = \mathbf{E}\,\mathbf{S}\,\mathbf{E}$ (26)	$S_{ik}' = S_{ik} e_i e_k = S_{ik} e^{-j\beta(l_i + l_k)}$ (27)
Mixed scattering matrix	$\mathbf{M}_{aa}' = \mathbf{E}_{aa}\mathbf{M}_{aa}\mathbf{E}_{aa}$ $\mathbf{M}_{ae}' = \mathbf{E}_{aa}\mathbf{M}_{ae}$ $\mathbf{M}_{ea}' = \mathbf{M}_{ea}\mathbf{E}_{aa}$ $\mathbf{M}_{ee}' = \mathbf{M}_{ee}$ (28)	$M_{ik}' = M_{ik}\, e_i\, e_k, \quad i,k \le m$ (29) $M_{ik}' = M_{ik}\, e_i, \quad i \le m, k > m$ $M_{ik}' = M_{ik}\, e_k, \quad i > m, k \le m$ $M_{ik}' = M_{ik}, \quad i > m, k > m$

Transformation Law

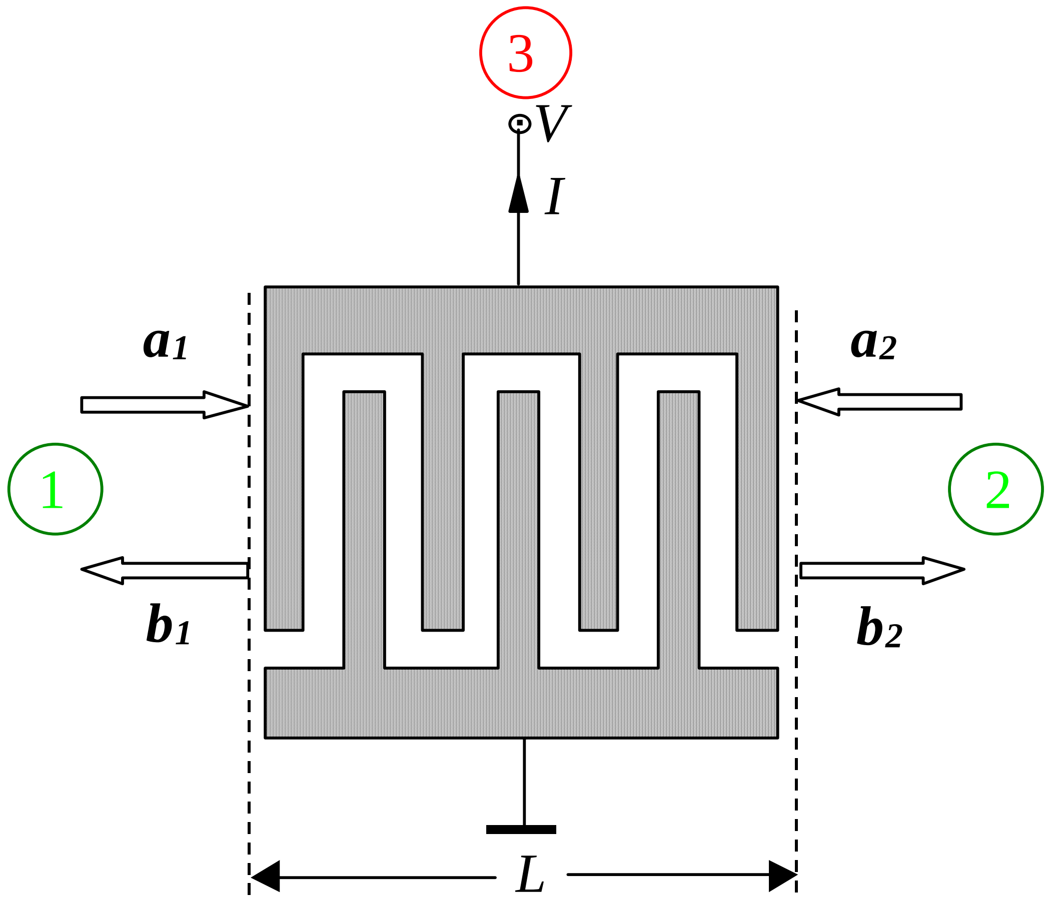
1. The reference plane shift at the *i*-th and *k*-th ports changes the phase of the scattering matrix element S_{ik} in accordance with the acoustic path change $l_i + l_k$ for the incident and reflected waves traveling throughout these ports.

2. It is only the elements of the mixed scattering matrix related to the acoustic ports which are subject to change due to reference plane transformation while elements referred to electrical ports remain unchanged.

3. Shifting the reference planes in the inward direction may be accounting for by reversing the exponent sign in Eqs. (24-29).

Part 2. SAW Transducer Mixed Scattering Matrix

Mixed Port Representation of a SAW Transducer



1, 2 – acoustic ports, 3 – electric port

Fig. 3. Three-port representation of a SAW transducer

Mixed Scattering Matrix of a SAW Transducer

An ideal SAW transducer is a reciprocal and lossless three-port acoustoelectric network with two acoustic and one electric ports.

Mixed scattering matrix of a SAW transducer

$$\begin{bmatrix} b_1 \\ b_2 \\ I \end{bmatrix} = \left[\begin{array}{cc|c} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ \hline m_{31} & m_{32} & m_{33} \end{array}\right] \begin{bmatrix} a_1 \\ a_2 \\ V \end{bmatrix} \quad (30)$$

Block-matrix form

$$\begin{bmatrix} \mathbf{B} \\ I \end{bmatrix} = \begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{ae}} \\ \mathbf{M_{ea}} & M_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ V \end{bmatrix} \quad (31)$$

where $\mathbf{A}=[a_1\ \ a_2]^T$ is the vector of the incident waves, $\mathbf{B}=[b_1\ \ b_2]^T$ is the vector of the reflected waves at the acoustic ports, I is the terminal current, V is the voltage applied to the transducer bus-bars at the electric port.

Physical Meaning of Matrix Blocks

Notation	Block	Type	Meaning	Mode
$\mathbf{M}_{aa}$	$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$	Acoustic	Scattering coeffcients of a short-circuit SAW transducer	Passive grating, V=0
$\mathbf{M}_{ae}$	$\begin{bmatrix} m_{13} & m_{23} \end{bmatrix}^T$	Acoustoelectric	Acoustoelectric conversion by a SAW transducer with the voltage *V* applied to the transducer bus-bars	SAW excitation, V≠0
$\mathbf{M}_{ea}$	$\begin{bmatrix} m_{31} & m_{32} \end{bmatrix}$	Electroacoustic	Terminal short-circuit current induced by the incident acoustic waves	SAW detection, *V*=0
$\mathbf{M}_{ee}$	m_{33}	Electric	Transducer admittance seen at the electric port when there are no incident waves at the acoustic ports	One-port electrical network V≠0

Mixed Scattering Matrix Elements

Element	Definition	Mode	Meaning	Units
m_{11}	b_1/a_1	Short-circuit, V=0, $a2$=0	Reflection coefficient of the short-circuit transducer at the left acoustic port 1	-
m_{22}	b_2/a_2	Short-circuit, V=0, $a1$=0	Reflection coefficient of the short-circuit transducer at the right acoustic port 2	-
m_{12}	b_1/a_2	Short-circuit, V=0, $a1$=0	Transmission coefficient of the short-circuit transducer in the left direction	-
m_{21}	b_2/a_1	Short-circuit, V=0, $a2$=0	Transmission coefficient of the short-circuit transducer in the right direction	-

Mixed Scattering Matrix Elements (Cont'd)

Element	Definition	Mode	Meaning	Units
m_{13}	b_1/V	SAW excitation, $a1=a2=0$, $V\neq 0$	Acoustoelectric conversion function, left direction (port 1)	$1/\sqrt{\Omega}$
m_{23}	b_2/V	SAW excitation, $a1=a2=0$, $V\neq 0$	Acoustoelectric conversion function, right direction (port 2)	$1/\sqrt{\Omega}$
m_{31}	I/a_1	SAW detection, short-circuit, $a2=0$, $V=0$	Electroacoustic conversion function from the left direction (current induced by the wave $a_1=1$ at the left acoustic port 1)	$1/\sqrt{\Omega}$
m_{32}	I/a_2	SAW detection, short-circuit, $a1=0$, $V=0$	Electroacoustic conversion function from the right direction (current induced by the wave $a_2=1$ at right acoustic port 2)	$1/\sqrt{\Omega}$
m_{33}	I/V	One-port electrical network, $a1=a2=0$, $V\neq 0$	Transducer admittance at the electric port	Ω^{-1}

Conversion of the Mixed Scattering Matrix to the Wave Scattering Matrix

Scattering matrix of a SAW transducer

$$\mathbf{S} = \begin{bmatrix} \mathbf{S_{aa}} & \mathbf{S_{ae}} \\ \mathbf{S_{ea}} & \mathbf{S_{ee}} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{aa} - \dfrac{\mathbf{M}_{ae}\mathbf{M}_{ea}}{Y_0 + Y} & 2\dfrac{\sqrt{Y_0}}{Y_0 + Y}\mathbf{M}_{ae} \\ -\dfrac{\sqrt{Y_0}}{Y_0 + Y}\mathbf{M}_{ea} & \dfrac{Y_0 - Y}{Y_0 + Y} \end{bmatrix} \tag{32}$$

or in the scalar form

$$\mathbf{S} = \begin{bmatrix} m_{11} - \dfrac{m_{13}m_{31}}{Y_0 + Y} & m_{12} - \dfrac{m_{13}m_{32}}{Y_0 + Y} & \dfrac{2\sqrt{Y_0}m_{13}}{Y_0 + Y} \\ m_{21} - \dfrac{m_{23}m_{31}}{Y_0 + Y} & m_{22} - \dfrac{m_{23}m_{32}}{Y_0 + Y} & \dfrac{2\sqrt{Y_0}m_{23}}{Y_0 + Y} \\ -\dfrac{\sqrt{Y_0}m_{31}}{Y_0 + Y} & -\dfrac{\sqrt{Y_0}m_{32}}{Y_0 + Y} & \dfrac{Y_0 - Y}{Y_0 + Y} \end{bmatrix} \tag{33}$$

where $Y_0=1/Z_0$ is the characteristic admittance at the electric port,
$Y=m_{33}$ is the transducer admittance.

Lossless Condition

For a lossless SAW transducer the wave scattering matrix **S** is unitary.

$$\mathbf{SS^*} = \begin{bmatrix} \mathbf{S}_{aa} & \mathbf{S}_{ae} \\ \mathbf{S}_{ea} & \mathbf{S}_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{aa} & \mathbf{S}_{ae} \\ \mathbf{S}_{ea} & \mathbf{S}_{ee} \end{bmatrix}^* = \begin{bmatrix} \mathbf{S}_{aa}\mathbf{S}_{aa}^* + \mathbf{S}_{ae}\mathbf{S}_{ae}^* & \mathbf{S}_{aa}\mathbf{S}_{ea}^* + \mathbf{S}_{ae}\mathbf{S}_{ee}^* \\ \mathbf{S}_{ea}\mathbf{S}_{aa}^* + \mathbf{S}_{ee}\mathbf{S}_{ae}^* & \mathbf{S}_{ea}\mathbf{S}_{ea}^* + \mathbf{S}_{ee}\mathbf{S}_{ee}^* \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0}^{\mathbf{T}} & 1 \end{bmatrix} \quad (34)$$

or

$$\begin{cases} \mathbf{S}_{aa}\mathbf{S}_{aa}^* + \mathbf{S}_{ae}\mathbf{S}_{ae}^* = \mathbf{E} \\ \mathbf{S}_{aa}\mathbf{S}_{ea}^* + \mathbf{S}_{ae}\mathbf{S}_{ee}^* = \mathbf{0} \\ \mathbf{S}_{ea}\mathbf{S}_{ea}^* + \mathbf{S}_{ee}\mathbf{S}_{ee}^* = \mathbf{1} \end{cases} \quad (35)$$

Properties of Reciprocal and Lossless SAW Transducer

Property	Equation	
	Matrix	Scalar
Reciprocity	$\mathbf{M}_{aa} = \mathbf{M}_{aa}^{\mathbf{T}}$	$m_{12} = m_{21}$
	$\mathbf{M}_{ea} = -2\mathbf{M}_{ae}^{\mathbf{T}}$	$m_{31} = -2\, m_{13}$ $m_{32} = -2\, m_{23}$
Power conservation	$\mathbf{M}_{aa}\mathbf{M}_{aa}^{*} = \mathbf{E}$	$\lvert m_{11}\rvert^2 + \lvert m_{12}\rvert^2 = 1$ $\lvert m_{22}\rvert^2 + \lvert m_{21}\rvert^2 = 1$ $m_{11}\, m_{12}^{*} + m_{22}^{*} m_{21} = 0$
	$\mathbf{M}_{ae} = \frac{1}{2}\mathbf{M}_{aa}\mathbf{M}_{ea}^{*}$	$m_{13} = -(\, m_{23}^{*} + m_{22}^{*}\, m_{23})\, / m_{21}^{*}$ $m_{23} = -(\, m_{13}^{*} + m_{11}^{*}\, m_{13})\, / m_{12}^{*}$
	$Re\left\{m_{33}\right\} = \mathbf{M}_{ae}^{*}\mathbf{M}_{ae}$	$Re\{m_{33}\} = \lvert m_{13}\rvert^2 + \lvert m_{23}\rvert^2$
Causality	$Im\{\mathbf{M_{ee}}\} = H\left\{\mathrm{Re}\{\mathbf{M_{ee}}\}\right\}$	$Im\{m_{33}\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Re\{m_{33}(\omega')\}}{\omega-\omega'}d\omega'$

Particular Case: $m_{11} = m_{22}, \quad m_{12} = m_{21} \Rightarrow \lvert m_{12}\rvert = \sqrt{1-\lvert m_{11}\rvert^2}, \theta_{11} = \theta_{12} \pm \frac{\pi}{2}$ (42)

Summary of SAW Transducer Properties

1. The mixed scattering matrix **M** of a SAW transducer contains three independent elements m_{11}, m_{13} , m_{33} to be determined, in general case.
2. The transmission coefficient m_{12} can be deduced from the reflection coefficient m_{11} using Eq. (41). The coefficients m_{11} and m_{12} are in the phase quadrature, with the phase ambiguity of π in general case.
3. Given the acoustoelectric conversion function m_{13} and the scattering coefficients m_{11} and m_{12}, the electroacoustic function m_{23} can be found by Eq. (39).
4. The electroacoustic conversion functions m_{31} and m_{32} can be found by reciprocity using Eqs. (37).
5. The real part (radiation conductance) $G(\omega)$ of the transducer admittance $Y(\omega)=G(\omega)+jB(\omega)+ j\omega C$ can be found from the power conservation low Eq. (40).
6. The imaginary part (radiation susceptance) $B(\omega)$ can be found by the Hilbert transformation of $G(\omega)$ (the causality principle).
7. The transducer static capacitance C should be found from the solution of the electrostatic problem in the closed-form or numerically.

Mixed Scattering and Transmission Matrices

The mixed transmission matrix **T**=[t_{ik}], *i,k*=1,2,3 describes the relationship of the acoustic waves a_1, b_1 at the left acoustic port 1 (as input) and the terminal current *I* with the waves a_2, b_2 at the right acoustic port 2 (as output) and the transducer bus-bar voltage *V*

$$\begin{bmatrix} a_1 \\ b_1 \\ I \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ V \end{bmatrix} \qquad (43)$$

Why Transmission Matrix?

The mixed transmission matrix **T** is appropriate for cascading SAW elements as it relates the waves at the input and output acoustic ports.

Conversion between Scattering and Transmission

	Scattering S	Transmission T
S	$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$	$\begin{bmatrix} \frac{t_{21}}{t_{11}} & t_{22}-\frac{t_{12}t_{21}}{t_{11}} & t_{23}-\frac{t_{21}t_{13}}{t_{11}} \\ \frac{1}{t_{11}} & -\frac{t_{12}}{t_{11}} & -\frac{t_{13}}{t_{11}} \\ \frac{t_{31}}{t_{11}} & t_{32}-\frac{t_{12}t_{31}}{t_{11}} & t_{33}-\frac{t_{13}t_{31}}{t_{11}} \end{bmatrix}$
T	$\begin{bmatrix} \frac{1}{m_{21}} & -\frac{m_{22}}{m_{21}} & -\frac{m_{23}}{m_{21}} \\ \frac{m_{11}}{m_{21}} & m_{12}-\frac{m_{11}m_{22}}{m_{21}} & m_{13}-\frac{m_{11}m_{23}}{m_{21}} \\ \frac{m_{31}}{m_{21}} & m_{32}-\frac{m_{22}m_{31}}{m_{21}} & m_{33}-\frac{m_{31}m_{23}}{m_{21}} \end{bmatrix}$	$\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$

Part 3. SAW Filter Simulation

SAW Filter Representation

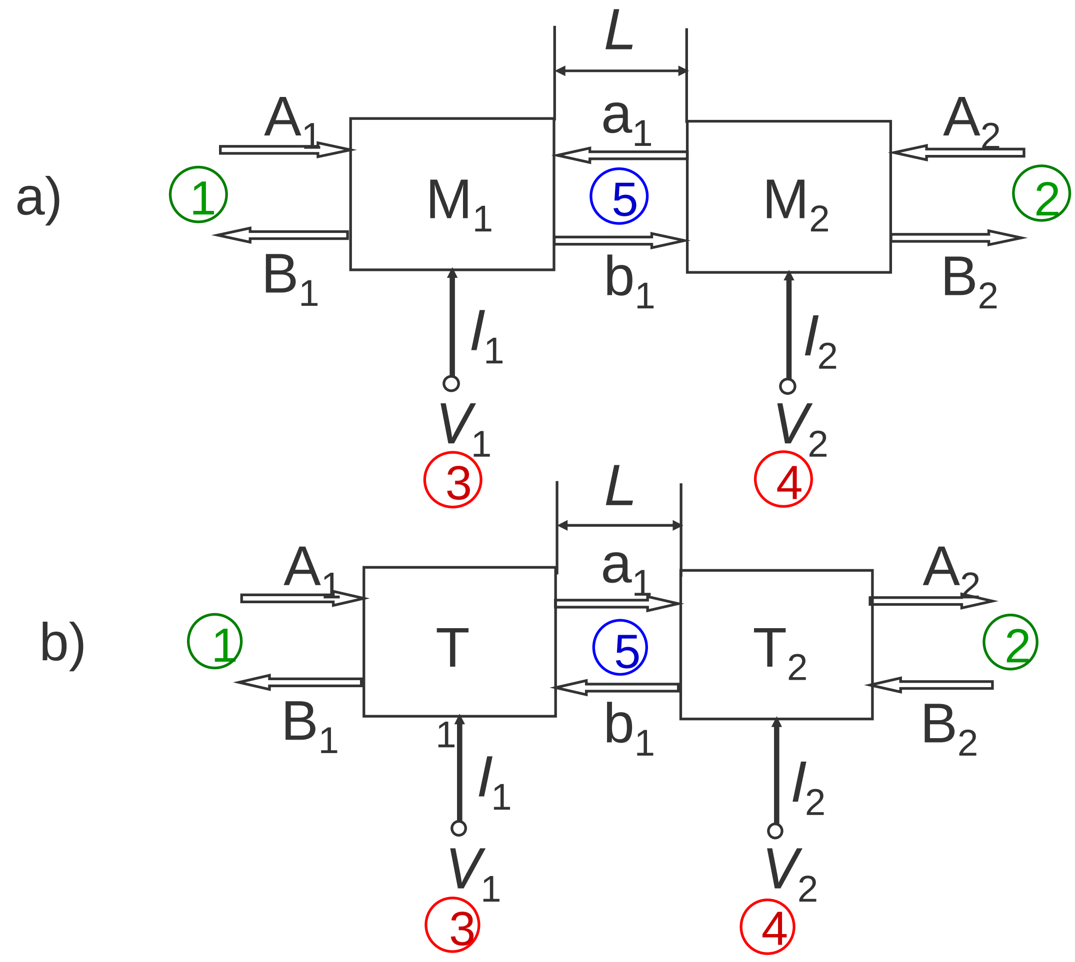


Fig. 4. Multi-port SAW filter representation in terms of:
a) mixed scattering matrices (**M**-matrices),
b) mixed transmission matrices (**T**-matrices)

Cascading Mixed Scattering Matrices

Mixed scattering matrix for each transducer

Input:

$$\begin{bmatrix} B_1 \\ b_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} m_{11}^1 & m_{12}^1 & m_{13}^1 \\ m_{21}^1 & m_{22}^1 & m_{23}^1 \\ m_{31}^1 & m_{32}^1 & m_{33}^1 \end{bmatrix} \begin{bmatrix} A_1 \\ a_1 \\ V_1 \end{bmatrix} \quad (44)$$

Output:

$$\begin{bmatrix} a_1 \\ B_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{12}^2 & m_{13}^2 \\ m_{21}^2 & m_{22}^2 & m_{23}^2 \\ m_{31}^2 & m_{32}^2 & m_{33}^2 \end{bmatrix} \begin{bmatrix} b_1 \\ A_2 \\ V_2 \end{bmatrix} \quad (45)$$

Overall Mixed Scattering Matrice

Total acoustoelectric system (SAW filter), no acoustic coupling conditions imposed

$$\begin{bmatrix} B_1 \\ B_2 \\ \hline I_1 \\ I_2 \\ \hline a_1 \\ b_1 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} m_{11}^1 & 0 & m_{13}^1 & 0 & m_{12}^1 & 0 \\ 0 & m_{22}^2 & 0 & m_{23}^2 & 0 & m_{21}^2 \\ \hline m_{31}^1 & 0 & m_{33}^1 & 0 & m_{32}^1 & 0 \\ 0 & m_{32}^2 & 0 & m_{33}^2 & 0 & m_{31}^2 \\ \hline 0 & m_{12}^2 & 0 & m_{13}^2 & 0 & m_{11}^2 \\ m_{21}^1 & 0 & m_{23}^1 & 0 & m_{22}^1 & 0 \end{array}\right] \begin{bmatrix} A_1 \\ A_2 \\ \hline V_1 \\ V_2 \\ \hline a_1 \\ b_1 \end{bmatrix} \tag{46}$$

Block-matrix form

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{I} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{V} \\ \mathbf{C} \end{bmatrix} \tag{47}$$

Nomenclature

A$=[A_1\ A_2]^T$	vector of the incident waves at the external acoustic ports 1, 2
B$=[B_1\ B_2]^T$	vector of the reflected waves at the external acoustic ports 1, 2
C$=[a_1\ b_1]^T$	vector of the traveling waves at the internal (coupled) acoustic ports 5
I $=[I_1\ I_2]^T$	vector of the currents at the electric ports 3, 4
V $=[V_1\ V_2]^T$	vector of the voltages at the electric ports 3, 4

General Solution

By excluding the unknown vector **C** we find the following solution of the matrix system of equations (47)

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{ae}} \\ \mathbf{M_{ea}} & \mathbf{M_{ee}} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{V} \end{bmatrix} \qquad (48)$$

where the mixed scattering matrix of a SAW filter

$$\mathbf{M} = \begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{ae}} \\ \mathbf{M_{ea}} & \mathbf{M_{ee}} \end{bmatrix} = \begin{bmatrix} \mathbf{M_{11}} & \mathbf{M_{12}} \\ \mathbf{M_{21}} & \mathbf{M_{22}} \end{bmatrix} + \begin{bmatrix} \mathbf{M_{13}} \\ \mathbf{M_{23}} \end{bmatrix} (\mathbf{E} - \mathbf{M_{33}})^{-1} \begin{bmatrix} \mathbf{M_{31}} & \mathbf{M_{32}} \end{bmatrix} \qquad (49)$$

$$\mathbf{M_{11}} = \begin{bmatrix} m_{11}^1 & 0 \\ 0 & m_{22}^2 \end{bmatrix}, \quad \mathbf{M_{12}} = \begin{bmatrix} m_{13}^1 & 0 \\ 0 & m_{23}^2 \end{bmatrix}, \quad \mathbf{M_{13}} = \begin{bmatrix} m_{12}^1 & 0 \\ 0 & m_{21}^2 \end{bmatrix}$$

$$\mathbf{M_{21}} = \begin{bmatrix} m_{31}^1 & 0 \\ 0 & m_{32}^2 \end{bmatrix}, \quad \mathbf{M_{22}} = \begin{bmatrix} m_{33}^1 & 0 \\ 0 & m_{33}^2 \end{bmatrix}, \quad \mathbf{M_{23}} = \begin{bmatrix} m_{32}^1 & 0 \\ 0 & m_{31}^2 \end{bmatrix}$$

$$\mathbf{M_{31}} = \begin{bmatrix} 0 & m_{12}^2 \\ m_{21}^1 & 0 \end{bmatrix}, \quad \mathbf{M_{32}} = \begin{bmatrix} 0 & m_{13}^2 \\ m_{23}^1 & 0 \end{bmatrix}, \quad \mathbf{M_{33}} = \begin{bmatrix} 0 & m_{11}^2 \\ m_{22}^1 & 0 \end{bmatrix}$$

Particular Case: Isolated Acoustoelectric System

There are no incident waves at the external acoustic ports *
(**A**=0, electrical two-port)

$$\mathbf{I} = \mathbf{Y}\mathbf{V} \tag{50}$$

where a system admittance matrix

$$\mathbf{Y} = \mathbf{M}_{ee} = \mathbf{M}_{22} + \mathbf{M}_{23}(\mathbf{E} - \mathbf{M}_{33})^{-1}\mathbf{M}_{32} = \mathbf{M}_{22} + \frac{1}{\Delta M}\mathbf{M}_{23}(\mathbf{E} + \mathbf{M}_{33})\mathbf{M}_{32} \tag{51}$$

$$\mathbf{Y} = \begin{bmatrix} m_{33}^1 + \frac{1}{\Delta M} m_{11}^2 m_{32}^1 m_{23}^1 & \frac{1}{\Delta M} m_{32}^1 m_{13}^2 \\ \frac{1}{\Delta M} m_{23}^1 m_{31}^2 & m_{33}^2 + \frac{1}{\Delta M} m_{22}^1 m_{31}^2 m_{13}^2 \end{bmatrix}, \tag{52}$$

$$\Delta M = 1 - m_{11}^2 m_{22}^1$$

Reflectionless SAW Transducer: Quasi-Static Approximation

In the quasi-static approximation (reflectionless SAW transducer)

$$m_{11}^{1,2} = m_{22}^{1,2} = 0.$$

Therefore,

$$\mathbf{Y} = \begin{bmatrix} m_{33}^{1} & m_{32}^{1} m_{13}^{2} \\ m_{23}^{1} m_{31}^{2} & m_{33}^{2} \end{bmatrix} = \begin{bmatrix} m_{33}^{1} & -2 m_{23}^{1} m_{13}^{2} \\ -2 m_{23}^{1} m_{13}^{2} & m_{33}^{2} \end{bmatrix} \qquad (53)$$

In the quasi-static approximation (reflectionless SAW transducers), the self-admittances Y_{11}, Y_{22} are defined by the admittance of the input and output SAW transducers, with the cross-admittance $Y_{12} = Y_{21}$ given by the product of the acoustoelectric functions of both transducers.

Cascading Mixed Transmission Matrices

Augmented transmission matrices of SAW transducers:

Input: *Output:*

$$\begin{bmatrix} A_1 \\ B_1 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} t_{11}^1 & t_{12}^1 & t_{13}^1 & 0 \\ t_{21}^1 & t_{22}^1 & t_{23}^1 & 0 \\ t_{31}^1 & t_{32}^1 & t_{33}^1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ V_1 \\ I_2 \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ b_1 \\ V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} t_{11}^2 & t_{12}^2 & 0 & t_{13}^2 \\ t_{21}^2 & t_{22}^2 & 0 & t_{23}^2 \\ 0 & 0 & 1 & 0 \\ t_{31}^2 & t_{32}^2 & 0 & t_{33}^2 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ V_1 \\ V_2 \end{bmatrix} \qquad (54)$$

where

$$\begin{bmatrix} t_{11}^{1,2} & t_{12}^{1,2} & t_{13}^{1,2} \\ t_{21}^{1,2} & t_{22}^{1,2} & t_{23}^{1,2} \\ t_{31}^{1,2} & t_{32}^{1,2} & t_{33}^{1,2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{m_{21}{}^{1,2}} & -\dfrac{m_{22}{}^{1,2}}{m_{21}{}^{1,2}} & -\dfrac{m_{23}{}^{1,2}}{m_{21}{}^{1,2}} \\ \dfrac{m_{11}{}^{1,2}}{m_{21}{}^{1,2}} & m_{12}{}^{1,2} - \dfrac{m_{11}{}^{1,2} m_{22}{}^{1,2}}{m_{21}} & m_{13}{}^{1,2} - \dfrac{m_{11}{}^{1,2} m_{23}{}^{1,2}}{m_{21}{}^{1,2}} \\ \dfrac{m_{31}{}^{1,2}}{m_{21}{}^{1,2}} & m_{32}{}^{1,2} - \dfrac{m_{22}{}^{1,2} m_{31}{}^{1,2}}{m_{21}{}^{1,2}} & m_{33}{}^{1,2} - \dfrac{m_{31}{}^{1,2} m_{23}{}^{1,2}}{m_{21}{}^{1,2}} \end{bmatrix} \qquad (55)$$

Overall Mixed Transmission Matrix

The overall mixed transmission matrix is given by the product of the augmented mixed transmission matrices

$$\begin{bmatrix} A_1 \\ B_1 \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & \mathbf{T_{13}} \\ T_{21} & T_{22} & \mathbf{T_{23}} \\ \mathbf{T_{31}} & \mathbf{T_{32}} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ \mathbf{V} \end{bmatrix} \tag{56}$$

Block-matrix form

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & \mathbf{T_{13}} \\ T_{21} & T_{22} & \mathbf{T_{23}} \\ \mathbf{T_{31}} & \mathbf{T_{32}} & \mathbf{T_{33}} \end{bmatrix} = \left[\begin{array}{c|c|cc} t_{11}^1 t_{11}^2 + t_{12}^1 t_{21}^2 & t_{11}^1 t_{12}^2 + t_{12}^1 t_{22}^2 & t_{13}^1 & t_{11}^1 t_{13}^2 + t_{12}^1 t_{23}^2 \\ \hline t_{21}^1 t_{11}^2 + t_{22}^1 t_{21}^2 & t_{21}^1 t_{12}^2 + t_{22}^1 t_{22}^2 & t_{23}^1 & t_{21}^1 t_{13}^2 + t_{22}^1 t_{23}^2 \\ \hline t_{31}^1 t_{11}^2 + t_{32}^1 t_{21}^2 & t_{31}^1 t_{12}^2 + t_{32}^1 t_{22}^2 & t_{33}^1 & t_{31}^1 t_{13}^2 + t_{32}^1 t_{23}^2 \\ t_{31}^2 & t_{32}^2 & 0 & t_{33}^2 \end{array}\right] \tag{57}$$

Particular Case: Isolated Acoustoelectric System

There are no incident waves at the external acoustic ports *
(A_1=B_2=0, electrical two-port)

$$\mathbf{Y} = \mathbf{T_{33}} - \mathbf{T_{31}} T_{11}^{-1} \mathbf{T_{13}} \tag{58}$$

Substitution of Eqs. (55), (57) into Rq. (58) gives the same result (52).

Brief Summary on Cascading Techniques

1. Two basic techniques for SAW transducer cascading are:
   - direct interconnection of the mixed scattering matrices
   - cascading the mixed transmission matrices.

2. Both techniques give the identical results.

3. Cascading techniques can be generalized to the case of *N* SAW transducers following similar guidelines as in the particular case of two in-line SAW transducers.

SAW Filter *S*-Parameters

Once two-port admittance matrix (*Y*-parameters) in the form of (51) or (58) has been determined, the scattering matrix **S** (*S*-parameters) can be found by using the matrix equation

$$\mathbf{S} = (\mathbf{Y_0} - \mathbf{Y})(\mathbf{Y_0} + \mathbf{Y})^{-1} \qquad (59)$$

$$\mathbf{Y_0} = \begin{bmatrix} Y_0 & 0 \\ 0 & Y_0 \end{bmatrix}$$

where $Y_0=1/Z_0$ is characteristic admittance (typically, $Z_0=50\ \Omega$)

Closed-form *S*-parameters

$$\mathbf{S} = \frac{1}{\Delta Y}\begin{bmatrix} (Y_0 - Y_{11})(Y_0 + Y_{22}) + Y_{12}Y_{21} & -2Y_{12}Y_0 \\ -2Y_{21}Y_0 & (Y_0 + Y_{11})(Y_0 - Y_{22}) + Y_{12}Y_{21} \end{bmatrix} \qquad (60)$$

$$\Delta Y = (Y_0 + Y_{11})(Y_0 + Y_{22}) - Y_{12}Y_{21}$$

The denominator ΔY accounts for the multiple reflections due to the regenerated signal (triple transit echo (TTE)) between the input and output SAW transducers loaded by the characteristic admittance Y_0 at each end.

Particular Case: High-Loss SAW Filter

Assumptions: $Y_{11} << Y_0$, $Y_{22} << Y_0$, and $Y_{12}Y_{21} << Y_0^2$

$$S_{12} = -2\frac{Y_{12}}{Y_0} = 4Z_0 m_{23}^1 m_{13}^2 \sim m_{23}^1 m_{13}^2 \qquad (61)$$

1. In general case, the SAW filter transfer function S_{12} is no longer proportional to the cross-admittance Y_{12} and therefore it has more complicated frequency behavior rather than just the product of the acoustoelectronic functions of the input and output SAW transducers (an idealized SAW filter frequency response).

2. This frequency response distortion must be accounted and compensated for at the SAW filter synthesis step.

Part 4. SAW Transducer Modeling in Quasi-Static Approximation

Mixed Scattering Matrix

Basic Assumption: A short-circuit SAW transducer is reflectionless, i.e. $m_{11} = m_{22} = 0$.

Practicality: Valid if the central frequency f_0 of a SAW transducer is far away from the synchronous frequency $f_\pi = v/2p$ where v is SAW velocity and p is the transducer period (pitch).

In the quasi-static approximation the mixed scattering matrix of a SAW transducer

$$\mathbf{M} = \begin{bmatrix} 0 & e^{-j\Phi} & m \\ e^{-\Phi} & 0 & -m^* e^{-j\Phi} \\ -2m & 2m^* e^{-j\Phi} & Y \end{bmatrix}, \quad \Phi = \beta L = \beta N p \qquad (62)$$

$\beta = \omega / v$ SAW wave number

$L = Np$ total acoustical length of a SAW transducer (port-to-port)

$m = m_{13}$ acoustoelectric function

Mixed Transmission Matrix

In the quasi-static approximation the mixed transmission matrix of a SAW transducer

$$\mathbf{T} = \begin{bmatrix} e^{j\Phi} & 0 & m^* \\ 0 & e^{-j\Phi} & m \\ -2me^{j\Phi} & 2m^*e^{-j\Phi} & \mathrm{Im}\{m_{33}\} \end{bmatrix} \tag{63}$$

$\beta = \omega / v$ SAW wave number

$L = Np$ total acoustical length of a SAW transducer (port-to-port)

$m = m_{13}$ acoustoelectric function

$\Phi = \beta L$ total phase lag

Phase Reference

The mixed scattering and transmission matrices take the simplest form when the phase is referenced to the transducer center (Φ=0):

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & m \\ 1 & 0 & -m^* \\ -2m & 2m^* & Y \end{bmatrix} \qquad (63)$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & m^* \\ 0 & 1 & m \\ -2m & 2m^* & \mathrm{Im}\{Y\} \end{bmatrix} \qquad (64)$$

The independent matrix elements are *m* and *Y* to be determined in the quasi-static approximation.

SAW Transducer Parameters

In the quasi-static approximation, the mixed scattering matrix of a SAW transducer is characterized by three independent parameters:

1) effective SAW velocity v
2) acoustoelectric conversion (transfer) function m
3) transducer admittance $Y=G+jB+j\omega C$ where C is the transducer static capacitance

The acoustic conductance G and susceptance B are interrelated via the Hilbert transform due to the causality principle.

Periodic SAW Transducer

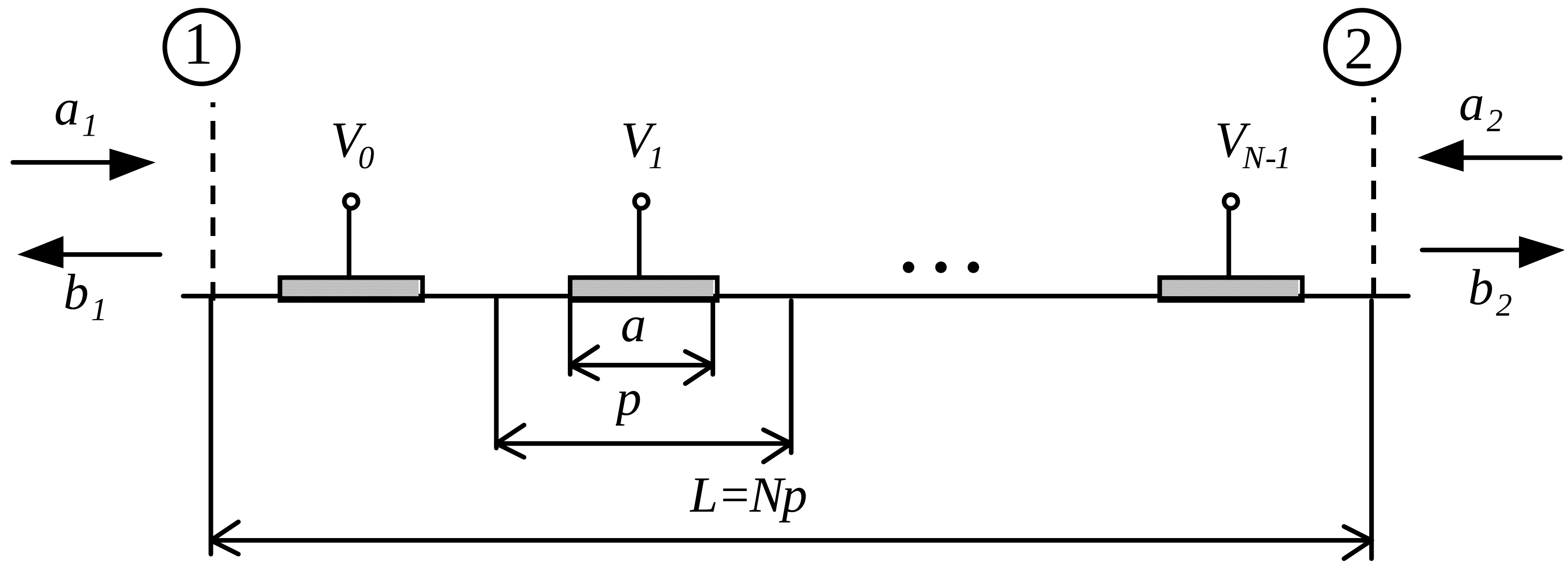


Fig. 5. Cross-section view of the finite length periodic SAW transducer

Acoustoelectric Conversion Function

The acoustoelectric conversion (transfer) function of a SAW transducer

$$m = \left.\frac{b_1}{V}\right|_{a_2=0} \qquad (65)$$

where b_1 is the generalized SAW amplitude of the wave traveling in the left direction, V is the voltage applied across the transducer.

SAW power flow carried by the uniform acoustic beam of the width W

$$P = \frac{1}{2}aa^* = \frac{1}{2}Y_0\phi\phi^* = \frac{1}{4}\frac{\omega W}{\Gamma}\phi\phi^* \qquad (66)$$

$a = \sqrt{Y_0}\phi$	generalized SAW amplitude
ϕ	surface potential accompanying a SAW of the amplitude a
$Y_0 = \omega W/2\Gamma$	acoustic characteristic admittance (after D.P.Morgan)
$\Gamma = K^2/2\varepsilon$	substrate material constant
K^2	piezoelectric coupling factor
$\varepsilon = \varepsilon_0 + \varepsilon_p$	surface effective permittivity
ε_0	permittivity of the medium above the substrate
ε_p	effective permittivity of the substrate

Closed-Form Equation (D.P.Morgan)

In the quasi-static approximation (D. P. Morgan), a potential ϕ of the surface acoustic wave launched by a periodic SAW transducer in the left direction is

$$\phi(\beta) = j\varepsilon\Gamma V\xi(\beta)F(\beta)e^{-j\beta L/2} \qquad (67)$$

$\beta=\omega/v$	SAW wave number
$\xi(\beta)$	element factor
$F(\beta)$	array factor
$L=Np$	transducer length
N	number of electrodes (fingers)
p	finger pitch (period).

An element factor $\xi(\omega)$ characterizes frequency behavior of the acoustic sources and depends on the metallization ratio.
An array factor $F(\omega)$ accounts for SAW filter selectivity and related with the finger polarity sequence by the Fourier transform.

Element Factor

In the quasi-static approximation, the element factor can be found from the electrostatic solution for the periodic strip array and takes the form

$$\xi(\nu) = \frac{2\sin\pi\nu}{P_{-\nu}(-\cos\Delta)} P_n(\cos\Delta) \qquad (68)$$

$\nu=\varphi/2\pi-n$	normalized frequency variable (base band)
$\varphi=\beta p$	phase lag per period *p*
$n=[\varphi/2\pi]$	space harmonic number $(n \le \varphi/2\pi \le n+1)$
$P_n(-\cos\Delta)$	Legendre polynomial
$P_{-\nu}(-\cos\Delta)$	Legendre function
$\Delta=\pi\eta$	dimensionless variable related to the electrode geometry
$\eta=a/p$	metallization ratio (duty factor)
a	finger width.

To the first order $\xi(\nu)\sim\sin\pi\nu$.

Array Factor

Fourier Transform Relation

The array factor $F(\varphi)$ is given by the Fourier transform of a set of the electrode potentials V_k, $k=0,N-1$:

$$F(\varphi)=\frac{1}{V}\sum_{k=0}^{N-1}V_k e^{-j(k-\frac{N-1}{2})\varphi}=\frac{1}{V}\sum_{k=0}^{N-1}V_k e^{-j\left\{(k+\frac{1}{2})\varphi-\frac{\Phi}{2}\right\}} \qquad (69)$$

$\Phi=\beta L=N\varphi$ is the total phase lag throughout the transducer, V is the voltage between transducer bus-bars.
The phase in Eq. (69) is referenced to the transducer center.

Acoustoelectric Function

In the quasi-static approximation the acoustoelectric function

$$m=\frac{a}{V}=\frac{\sqrt{Y_0}\phi}{V}=\frac{j}{2}\sqrt{\omega W K^2\varepsilon}\,\xi(\varphi)F(\varphi)e^{-j\Phi/2} \qquad (70)$$

Unapodized Periodic SAW Transducers

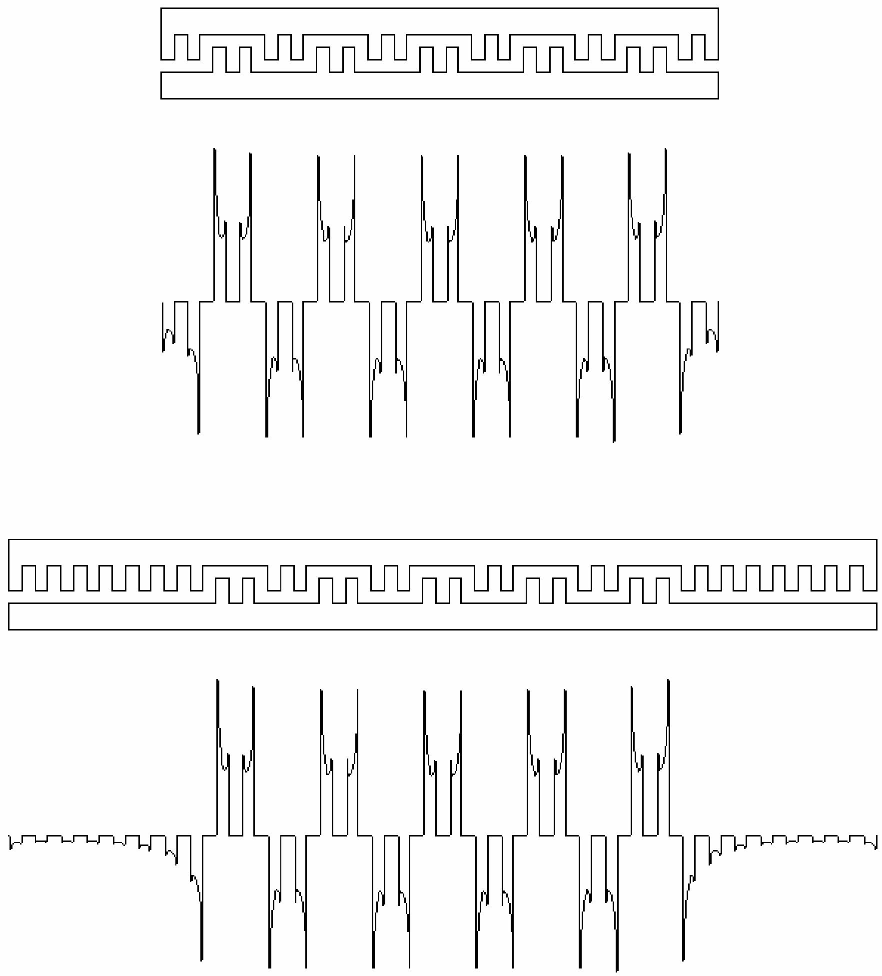


Fig. 6. Charge distribution in the finite length periodic SAW transducer

Basic Structure and Guard Electrodes

Guard electrodes are introduced in the finite length SAW transducer to suppress electrostatic end effects.

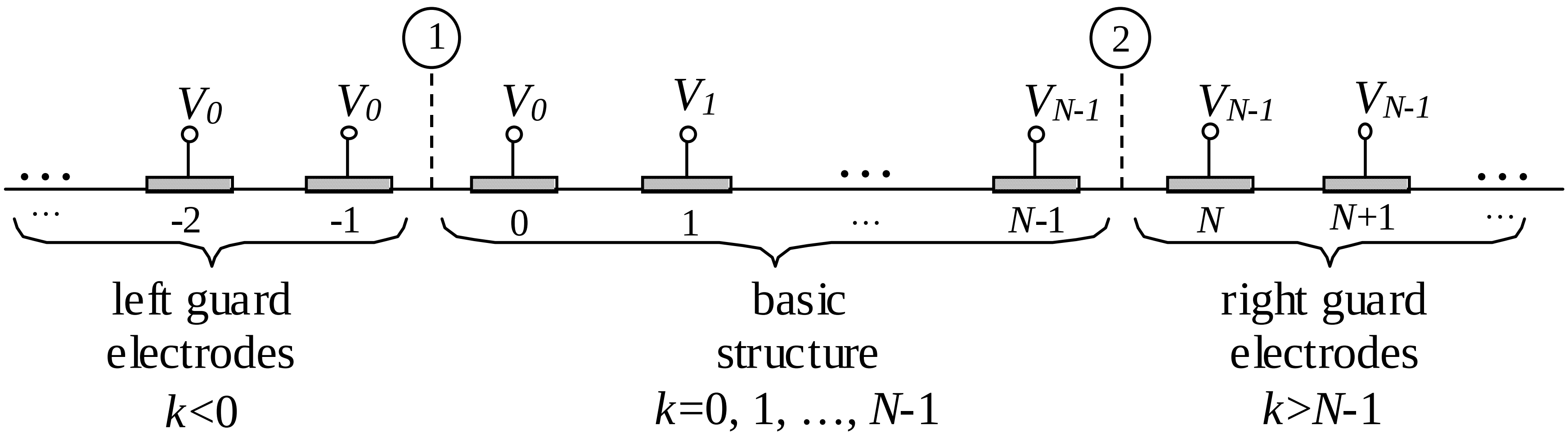


Fig. 7. Cross-section view of the periodic SAW transducer with guard electrodes

Contribution of Guard Electrodes to Acoustoelectric Conversion (Particular Case)

Potentials on the guard electrodes

$$V_k = \begin{cases} V_0, & k < 0 \\ V_{N-1}, & k > N-1 \end{cases}$$ left guard electrodes / right guard electrodes (70)

Particular case: all equipotential (grounded) guard fingers $V_0 = V_{N-1} = 0$

$$F(\varphi) = \frac{1}{V} \sum_{k=-\infty}^{\infty} V_k e^{-jk\varphi} = \frac{1}{V} \sum_{k=0}^{N-1} V_k e^{-jk\varphi} \qquad (71)$$

The finite summation in Eq. (71) gives the correct result for the infinite structure including guard electrodes contribution.

Contribution of Guard Electrodes to Acoustoelectric Conversion (General Case)

General case: Non-equipotential guard electrodes at the ends $V_0 \neq V_{N\text{-}1}$

$$\sum_{k=-\infty}^{\infty} V_k e^{-jk\varphi} = \overset{(left\ guard)}{V_0 \sum_{k=-\infty}^{-1} e^{-jk\varphi}} + \overset{(basic)}{\sum_{k=0}^{N-1} V_k e^{-jk\varphi}} + \overset{(right\ guard)}{V_{N-1} \sum_{k=N}^{\infty} e^{-jk\varphi}} =$$

$$= \sum_{k=0}^{N-1} V_k e^{-jk\varphi} + \frac{V_{N-1} e^{-jN\varphi} - V_0}{1 - e^{-j\varphi}} = \sum_{k=0}^{N-1} (V_k - V_{N-1}) e^{-jk\varphi} + \frac{V_{N-1} - V_0}{1 - e^{-j\varphi}} \qquad (72)$$

where the following identities have been used

$$\sum_{k=0}^{N-1} e^{\pm jk\varphi} = \frac{1 - e^{\pm jN\varphi}}{1 - e^{\pm j\varphi}}, \quad \sum_{k=0}^{\infty} e^{\pm jk\varphi} = \lim_{N\to\infty} \sum_{k=0}^{N-1} e^{\pm jk\varphi} = \frac{1}{1 - e^{\pm j\varphi}}$$

$$F(\varphi) = \frac{1}{V} \left\{ \sum_{k=0}^{N-1} (V_k - V_{N-1}) e^{-jk\varphi} + \frac{V_{N-1} - V_0}{1 - e^{-j\varphi}} \right\} \qquad (73)$$

In the particular case of the grounded guard fingers $V_0 = V_{N-1} = 0$, Eq. (73) reduces to Eq. (71).

Finger and Gap Taps

The sum in Eq. (73) can be transformed to another form

$$\sum_{k=-\infty}^{\infty} V_k e^{-jk\varphi} = \frac{1}{1-e^{-j\varphi}}\left(\sum_{k=1}^{N} V_k e^{-jk\varphi} - \sum_{k=0}^{N-1} V_k e^{-j(k+1)\varphi}\right) = \frac{1}{2j\sin\varphi/2}\sum_{k=0}^{M-1} \Delta V_k e^{-j(k+\frac{1}{2})\varphi} \quad (74)$$

where $\Delta V_k = V_{k+1} - V_k$ is the voltage in the *k*-th gap between two adjacent fingers having the potentials V_{k+1} and V_k, respectively, $M=N-1$ is number of the gaps in the basic sructure.

As the last gap voltage in Eq. (74) $\Delta V_{N-1}=V_N-V_{N-1}=0$, the summation corresponds to the number of gaps *M* in the basic structure. The factor $e^{-j\varphi/2}$ accounts for the gap position offset with respect to the finger center.

According to Eq. (74), the array factor can be alternatively expressed in terms of the Fourier transform of the gap voltages ΔV_k that gives zero contribution of the guard fingers to the overall response regardless a set of the potentials V_k.

Finger and Gap Taps Properties

1. Finger taps are located in the electrode centers, with amplitudes proportional to the electrode potentials V_k.

2. Gap taps are attributed to the interelectrode gaps, with amplitudes proportional to the gap voltages $\Delta V_k = V_{k+1} - V_k$.

3. Using the voltages ΔV_k instead of the potentials V_k excludes implicitly a uniform potential applied across the transducer.

4. Both finger and gap taps give exactly the same results if applied correctly. However, a special care must be taken while using in modeling finger taps.

Acoustoelectric Function (Gap Taps)

The sum in Eq. (73) can be transformed to another form

$$\sum_{k=-\infty}^{\infty} V_k e^{-jk\varphi} = \frac{1}{1-e^{-j\varphi}}\left(\sum_{k=1}^{N} V_k e^{-jk\varphi} - \sum_{k=0}^{N-1} V_k e^{-j(k+1)\varphi}\right) = \frac{1}{2j\sin\varphi/2}\sum_{k=0}^{M-1} \Delta V_k e^{-j(k+\frac{1}{2})\varphi} \qquad (74)$$

The acoustoelectric conversion function in terms of gap voltages

$$m = \frac{1}{2}\sqrt{\omega W K^2 \varepsilon}\,\zeta(\varphi)\Delta\tilde{F}(\varphi)e^{-j\Phi/2} \qquad (75)$$

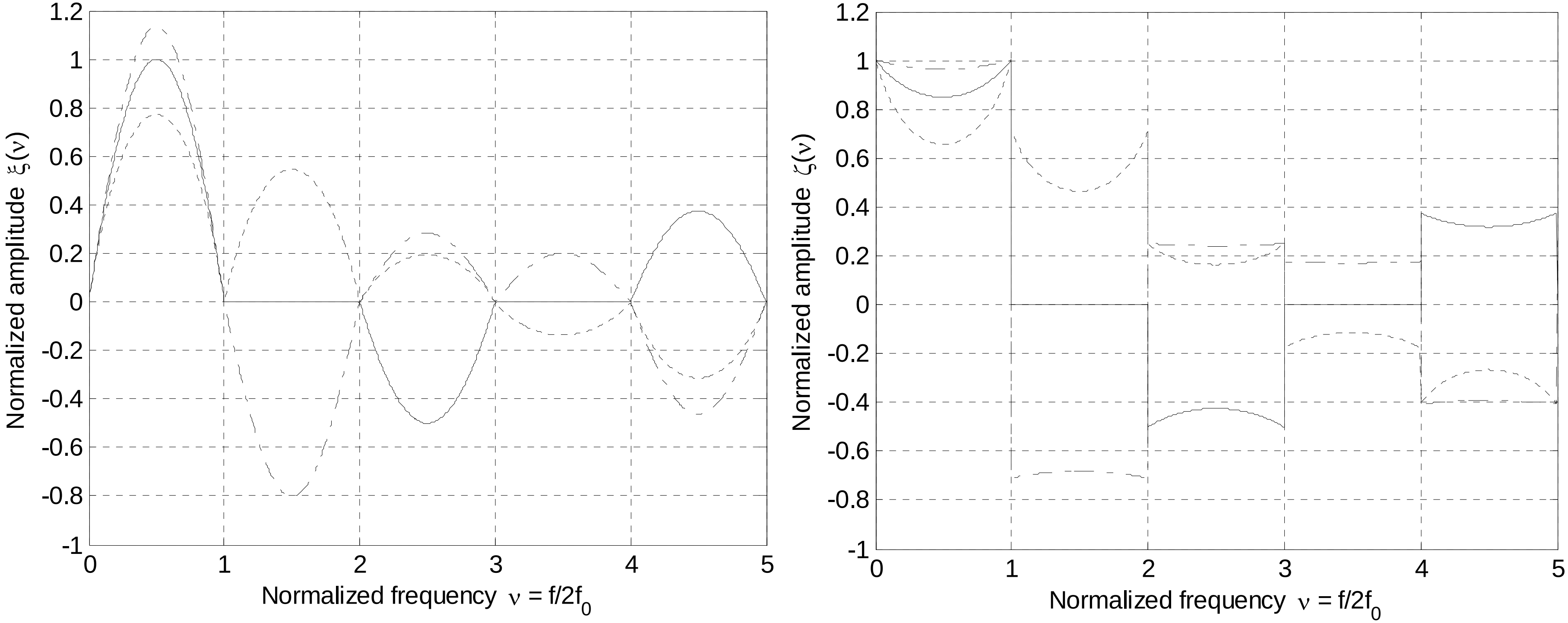
where the gap element factor $\zeta(\varphi)$

$$\zeta(\varphi) = \frac{\xi(\varphi)}{2\sin\varphi/2} = \frac{P_n(\cos\Delta)}{P_{-\nu}(-\cos\Delta)}\,,\ \ \nu = \varphi/2\pi. \qquad (76)$$

and the gap array factor

$$\tilde{F}(\varphi) = 2j\sin\varphi/2F(\varphi) = \sum_{k=0}^{M-1} \Delta V_k e^{-j(k+\frac{1}{2})\varphi} \qquad (77)$$

Element Factor Properties



a) finger element factor $\xi(\nu)$

b) gap element factor $\zeta(\nu)$

Fig. 7. Normalized element factor at first five space harmonics
for different metallization ratios:
- - - η=0.25, — η= 0.5, – - – η= 0.75

Misusing Finger Taps

Problem: What is the passband width of the solid (unsplit) finger unapodized SAW transducer with the number of fingers *N* and the central frequency f_0?

Solution 1 (wrong):

In terms of the finger taps $V_k=(-1)^k$, the frequency response

$$F(\varphi)=\sum_{k=0}^{N-1}(-1)^k e^{-jk\varphi}=\sum_{k=0}^{N-1}e^{-jk(\varphi-\pi)}=e^{-j\frac{N-1}{2}\varphi}\frac{\sin\frac{N}{2}(\varphi-\pi)}{\sin\frac{1}{2}(\varphi-\pi)},\quad \varphi=\pi\frac{f}{f_0}.$$

The main lobe (pass band) width

$$\Delta F=2\Delta f\text{, where }\Delta f=\frac{2f_0}{N}.$$

Solution 2 (correct):

In terms of the gap taps, we can find the correct result as

$$\Delta F=2\Delta f\text{, where }\Delta f=\frac{2f_0}{M}=\frac{2f_0}{N-1}.$$

i.e. the main lobe width is proportional to $1/M > 1/N$ where $M=N-1$ is the number of gaps in the transducer.

Merits of Gap Taps Against Finger Taps

1. Both models are theoretically equivalent if correctly applied. However, in general case the finger taps model is awkward to account for a contribution of the guard electrodes to the overall frequency response requiring more sophisticated equations.

2. Contrary to finger taps, the gap taps model gives identically a zero contribution of the guard electrodes to the overall response, regardless a set of the potentials V_k in the basic structure.

3. The gap taps model comprises the finite summation over the gaps in the basic structure, with the total number of gaps $M=N-1$.

4. Since we use the gap voltages ΔV_k instead of the potentials V_k, this excludes any uniform potential applied across the transducer while in the finger taps model this uniform potential must be implicitly taken into consideration and included into equations.

Merits of Gap Taps Against Finger Taps (Cont'd)

5. Some textbooks refer to wrong equations in terms of the of the finger taps model which are valid in the particular cases only.

6. The gap element factor has the simpler shape and weaker (flatter) frequency dependence if compared to the finger element factor.

7. It is highly recommended to use the gap taps model rather than the finger taps model wherever possible as misusing the finger taps model may cause troubleshooting in SAW filter simulation, in particular, for wideband SAW filters.

Apodized Periodic SAW Transducers

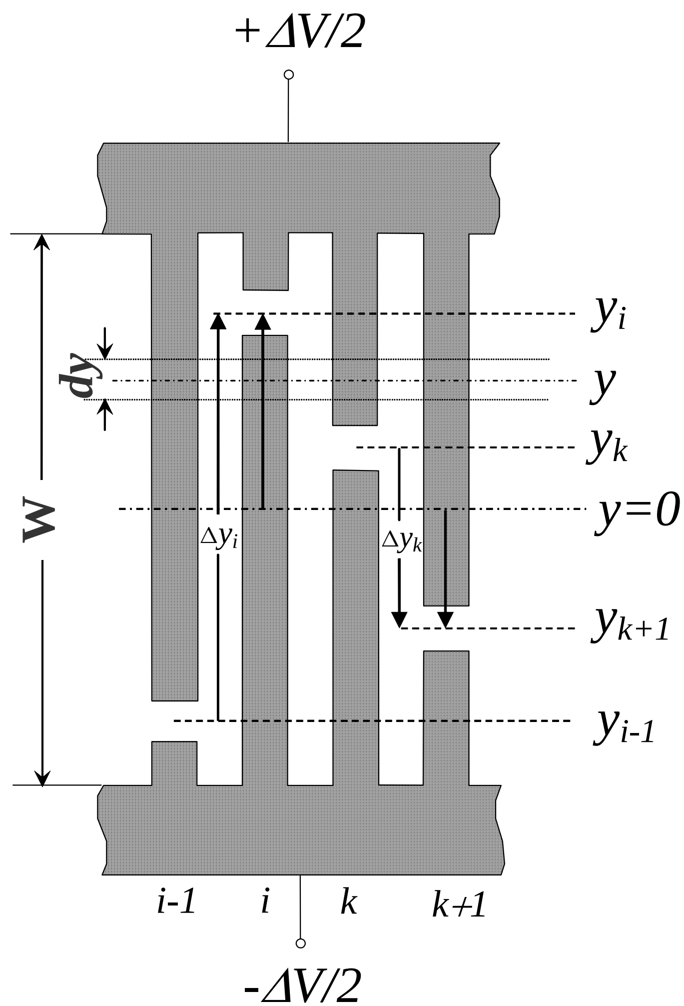


Fig. 8. Finger and gap tap weights of the apodized SAW transducer

Basic Assumptions

1. The propagating wavefront launched by an apodized SAW transducer is intercepted by the uniform receiving SAW transducer or multistrip coupler.
2. Each electrode is connected to either of two parallel bus-bars and complemented by a dummy finger for the wave front equalization.
3. Transversal electrostatic end effects are neglected.
4. The diffraction and beam steering effects are ignored.

A detected signal is not affected if the actual two-dimensional distribution $\phi(x,y)$ of the surface wave potential is replaced by the averaged distribution over the acoustic aperture *W.*

$$\overline{\phi}(x) = \frac{1}{W} \int_{-W/2}^{W/2} \phi(x,y)dy \tag{78}$$

Generalization of Finger Taps

Fourier transform of Eq. (78) gives

$$\bar{\phi}(\beta)=\frac{1}{W}\int_{-W/2}^{W/2}\phi(\beta,y)dy=j\varepsilon V\xi(\beta)\bar{F}(\beta)e^{-j\beta L/2} \quad (79)$$

In terms of the finger taps the averaged array factor

$$\bar{F}(\varphi)=\frac{1}{W}\int_{-W/2}^{W/2}F(\varphi,y)dy=\frac{1}{\Delta V}\sum_{k=0}^{N-1}\bar{V}_k e^{-j(k-\frac{N-1}{2})\varphi} \quad (80)$$

where ΔV is the voltage applied to the transducer bus-bars.

To determine the averaged potentials $\bar{V}_k$ across the aperture W, we consider a structure with a set of finger potentials $V_k=\pm\Delta V/2$.

The mean potential averaged across the aperture W can be found as

$$\bar{V}_k=\frac{1}{W}\int_{-W/2}^{W/2}V_k(y)dy=\frac{1}{W}\int_{-W/2}^{y_k}V_k(y)dy+\frac{1}{W}\int_{y_k}^{W/2}V_k(y)dy=-\frac{y_k}{W}\Delta V \quad (81)$$

Generalization of Gap Taps

From Eq. (81) the mean gap voltage is given by

$$\Delta\overline{V}_k = \overline{V}_{k+1} - \overline{V}_k = -\frac{\Delta y_k}{W}\Delta V \qquad (82)$$

where $\Delta y_k = y_{k+1} - y_k$ is the overlap of the adjacent fingers with the transversal gap positions y_{k+1} and y_k, respectively.

1. The effective finger and gap taps $\overline{V}_k$ and $\Delta\overline{V}_k$ are essentially fractions of the bus-bar voltage ΔV weighted by the normalized transversal gap positions y_k / W or finger overlaps $\Delta y_k / W$, respectively.
2. For apodized SAW transducers, the acoustoelectric conversion function can be found in terms of the effective tap weights $\overline{V}_k$ and $\Delta\overline{V}_k$ replacing the conventional potentials V_k or gap voltages ΔV_k.

Example of SAW Filter Simulation

Specifications

Central frequency f_0	70 MHz
Pass band width at -3 dB	9.4 MHz
Pass band width at -40 dB	10.6 MHz
Shape factor (-40/-3 dB)	1.13
Pass band ripple (peak-to-peak)	0.5 dB
Stop band attenuation	-50 dB
Insertion loss (matched)	25 dB

Example of SAW Filter Simulation

Design Parameters

Synchronous frequency	$f_{\pi}=2f_{0}$
Number of fingers per wave length	4 (split fingers)
Number of fingers (unapodized IDT)	48
Number of fingers (apodized IDT)	700
Acoustic aperture	2.5 mm (≈53 λ)
Substrate material	112º $LiTaO_3$
Die size	11.9 x 3.1 mm

Frequency Response

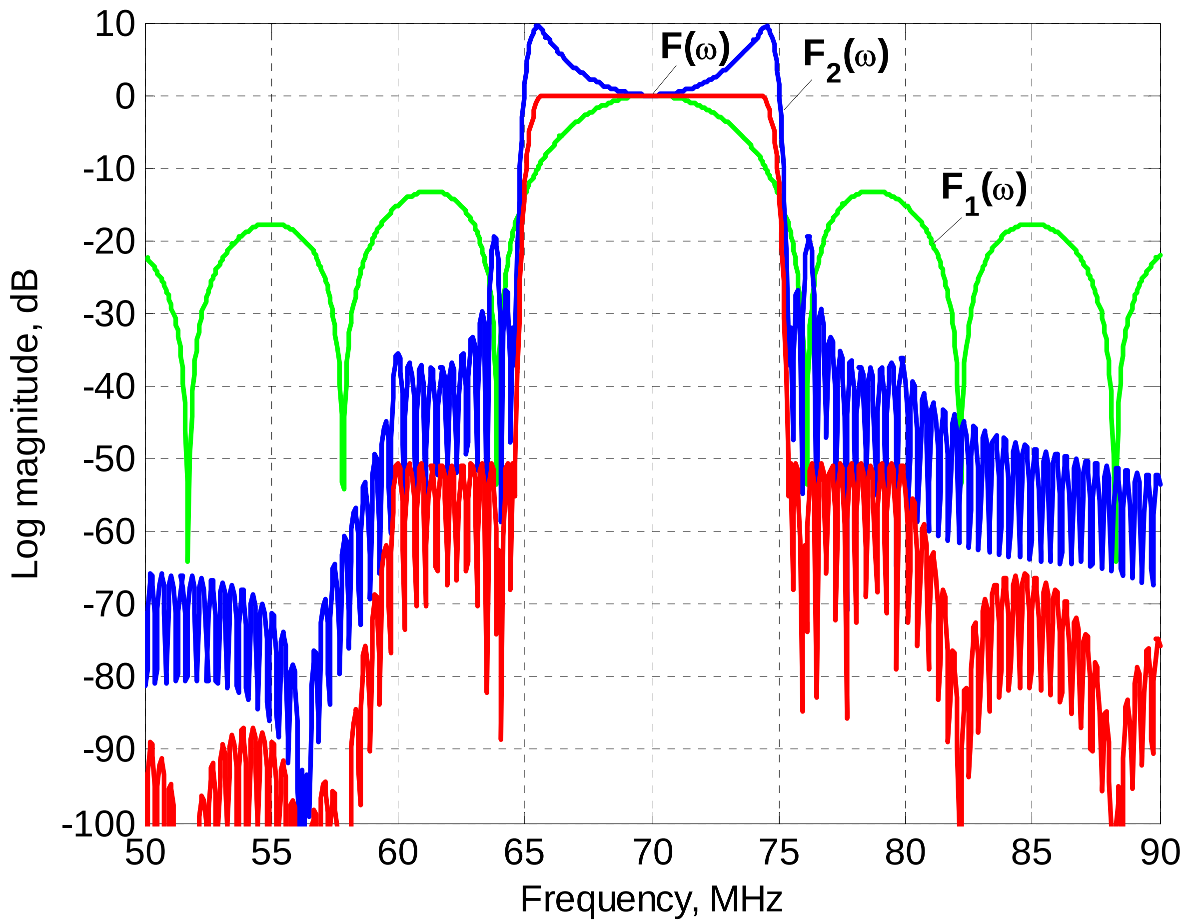


Fig. 9. SAW filter ideal frequency response:

*F*1(ω) – input, *F*2(ω) – output, *F*(ω) =*F*1(ω) *F*2(ω) - SAW filter

Time Response

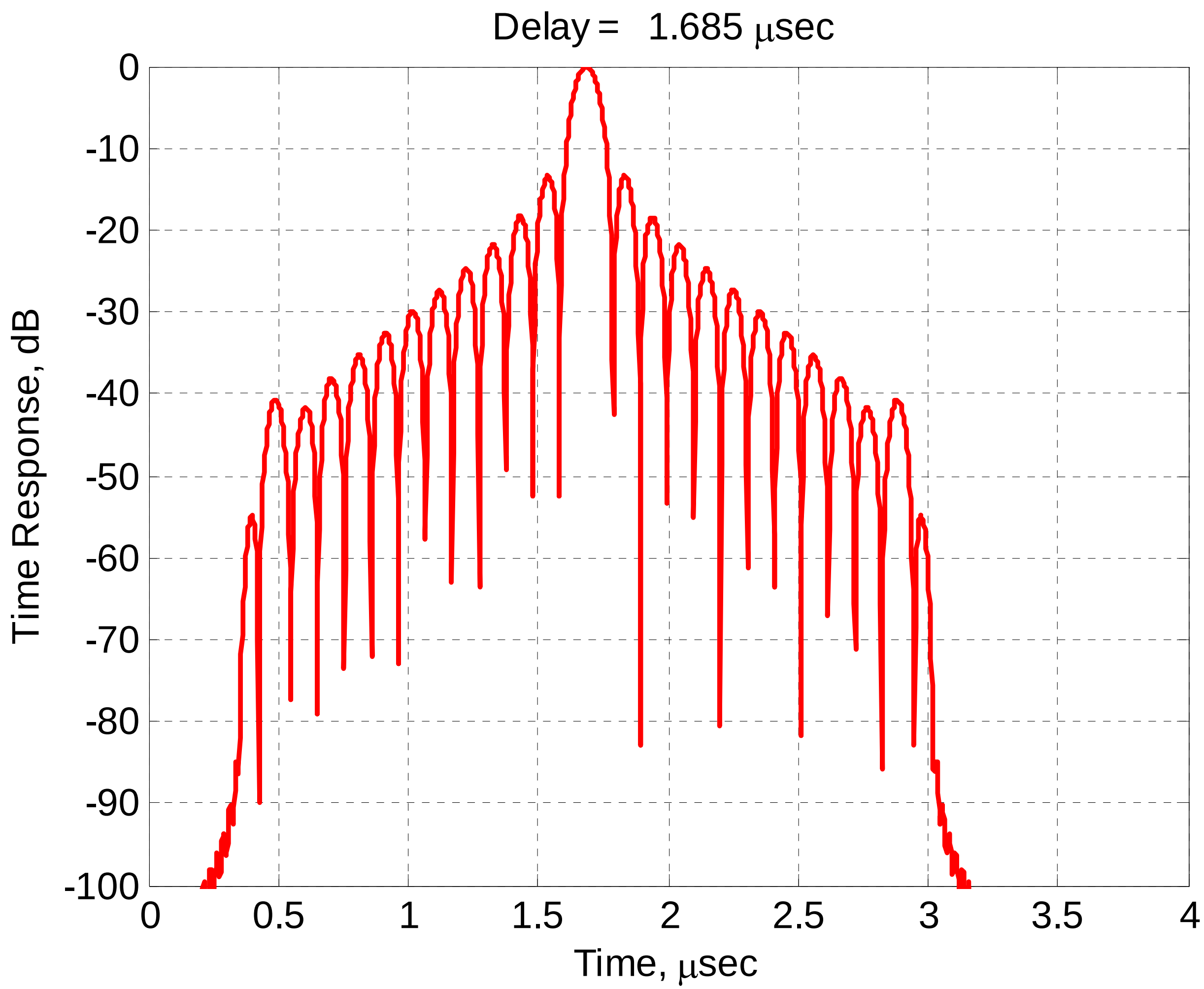
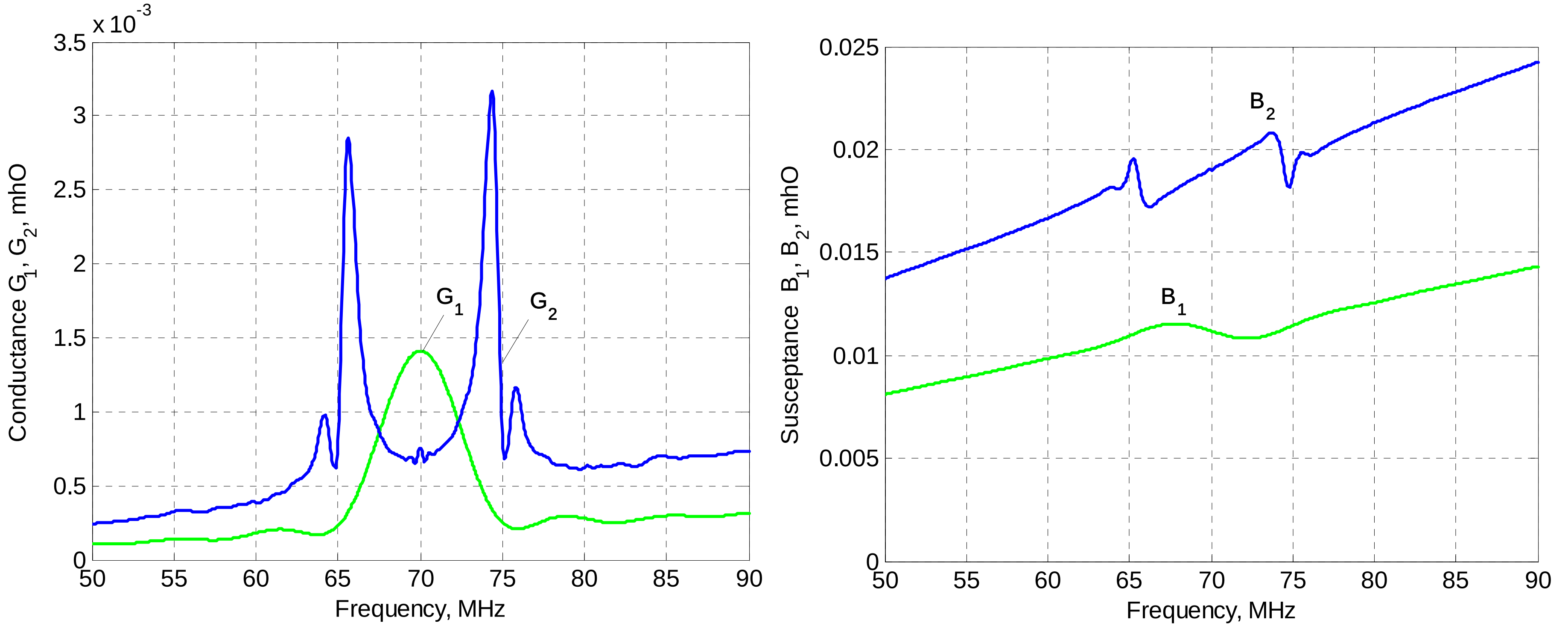


Fig. 10. SAW filter ideal impulse response

Input/Output Admittance

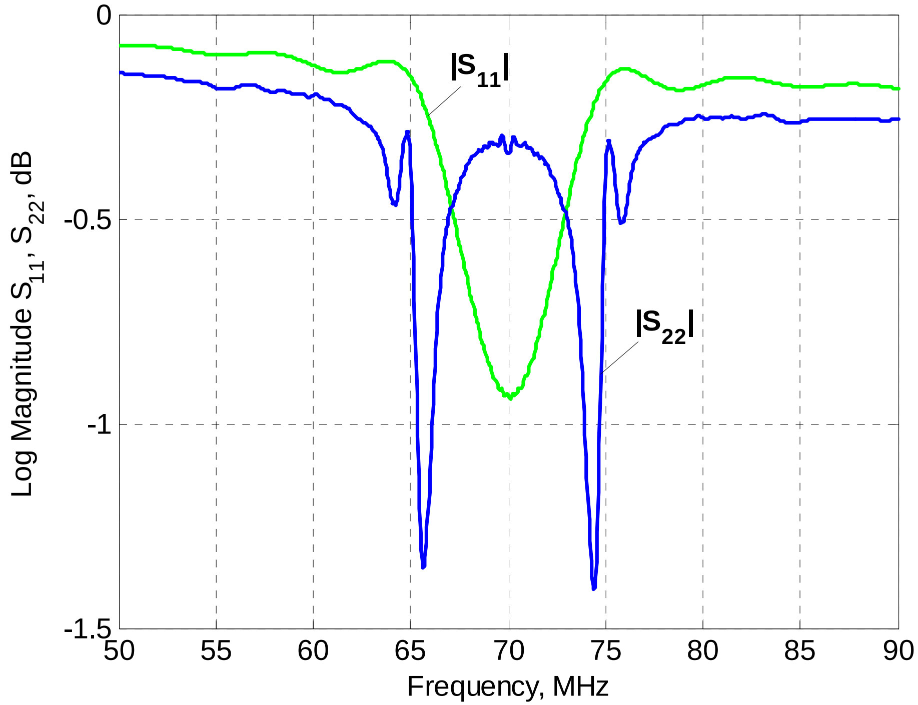


a) conductance

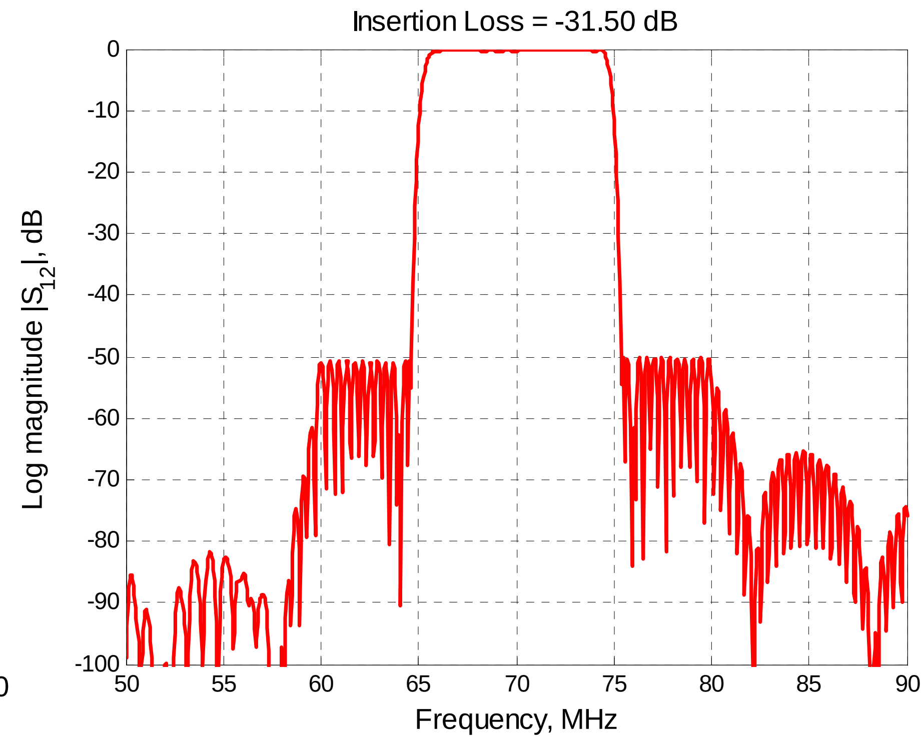
b) susceptance

Fig. 11. SAW filter admittance $Y_{1,2}(\omega)=G_{1,2}(\omega)+jB_{1,2}(\omega)$:
1 - input, 2 - output

S-Parameters (Unmatched)



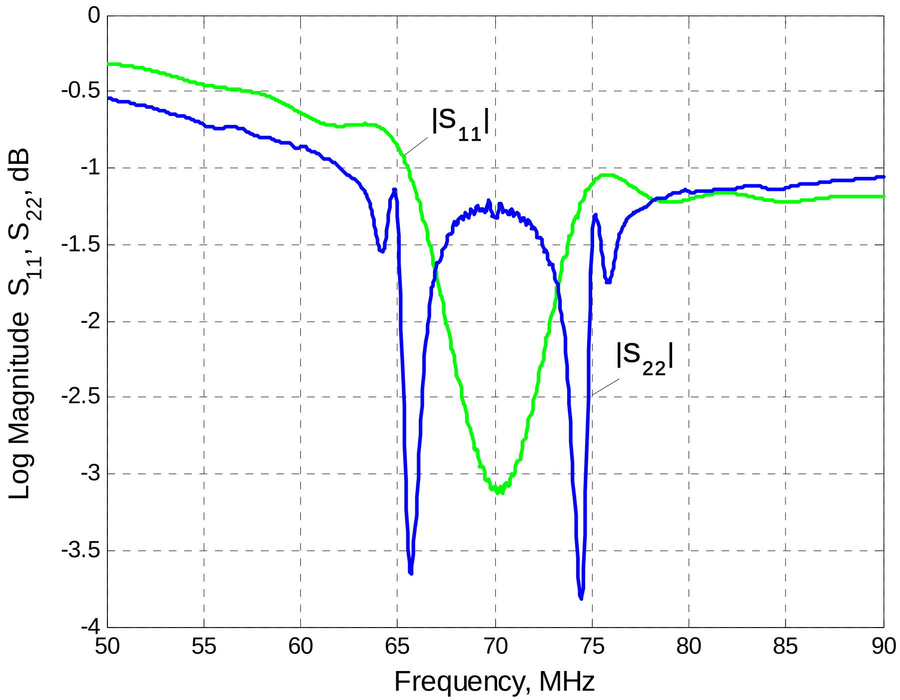
a) reflection coefficient S_{11}, S_{22}



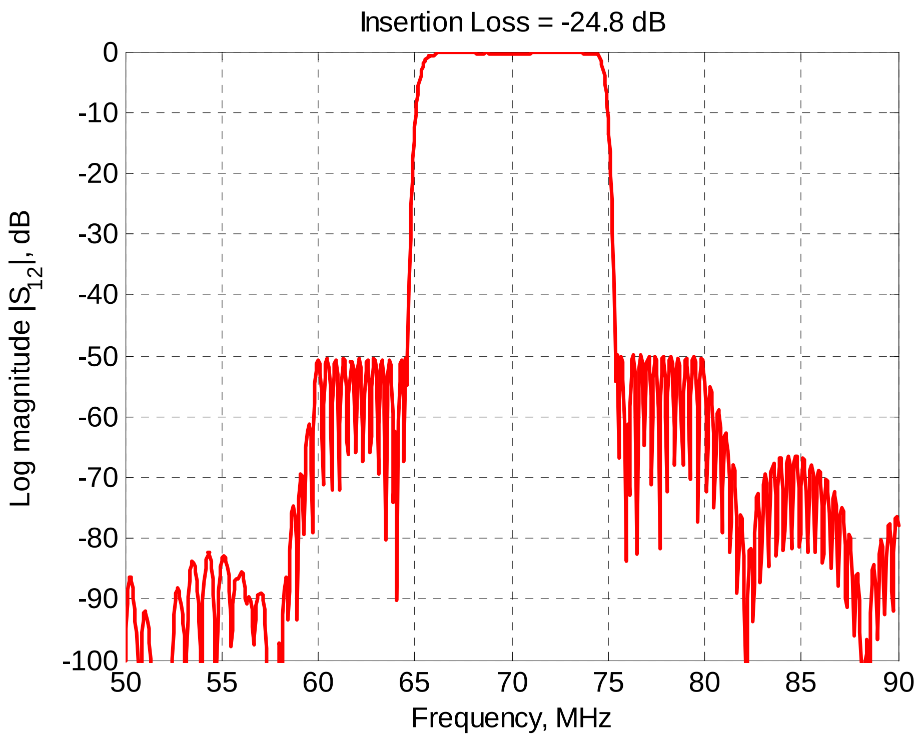
b) transmission coefficient $S_{12} = S_{21}$

Fig. 12. SAW filter S-parameters (unmatched)

S-Parameters (Matched)



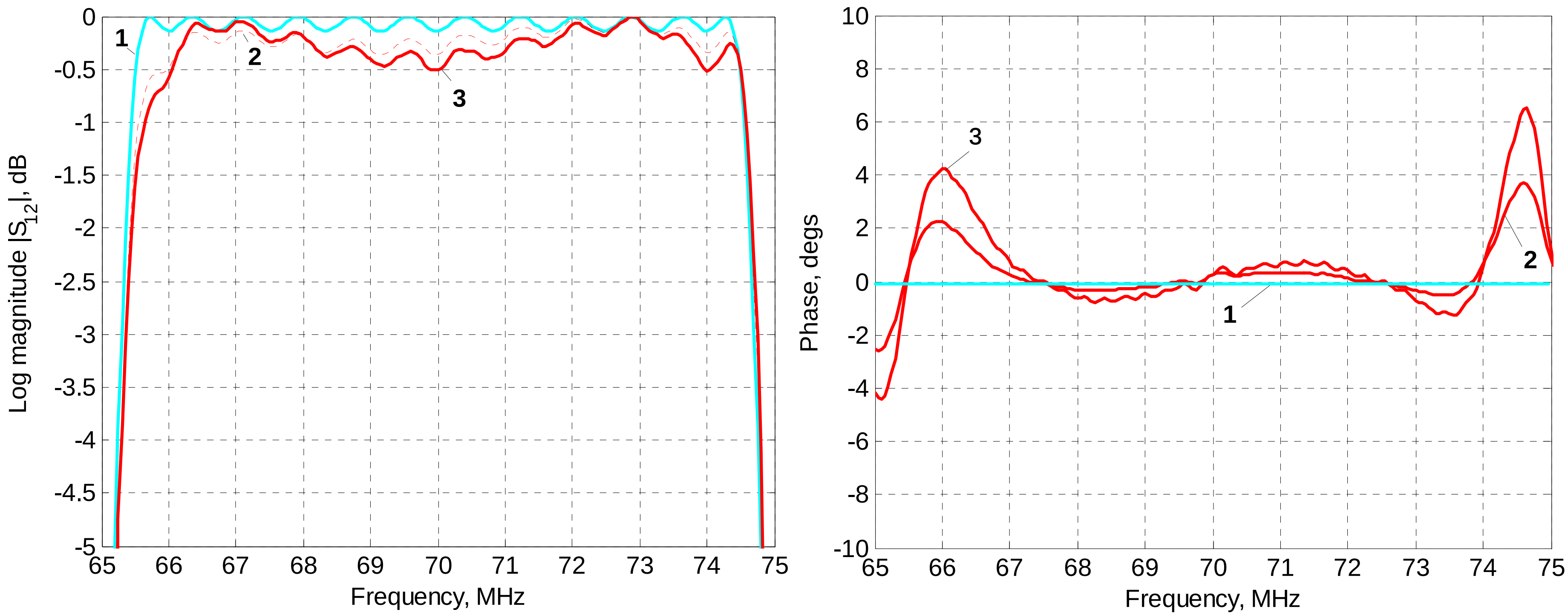
a) reflection coefficient S_{11}, S_{22}



b) transmission coefficient $S_{12} = S_{21}$

Fig. 13. SAW filter S-parameters (matched)

Pass Band Response



a) magnitude

b) phase

Fig. 14. SAW filter passband response without distortion compensation: 1 – ideal, 2 – unmatched, 3 - matched

Pass Band Response

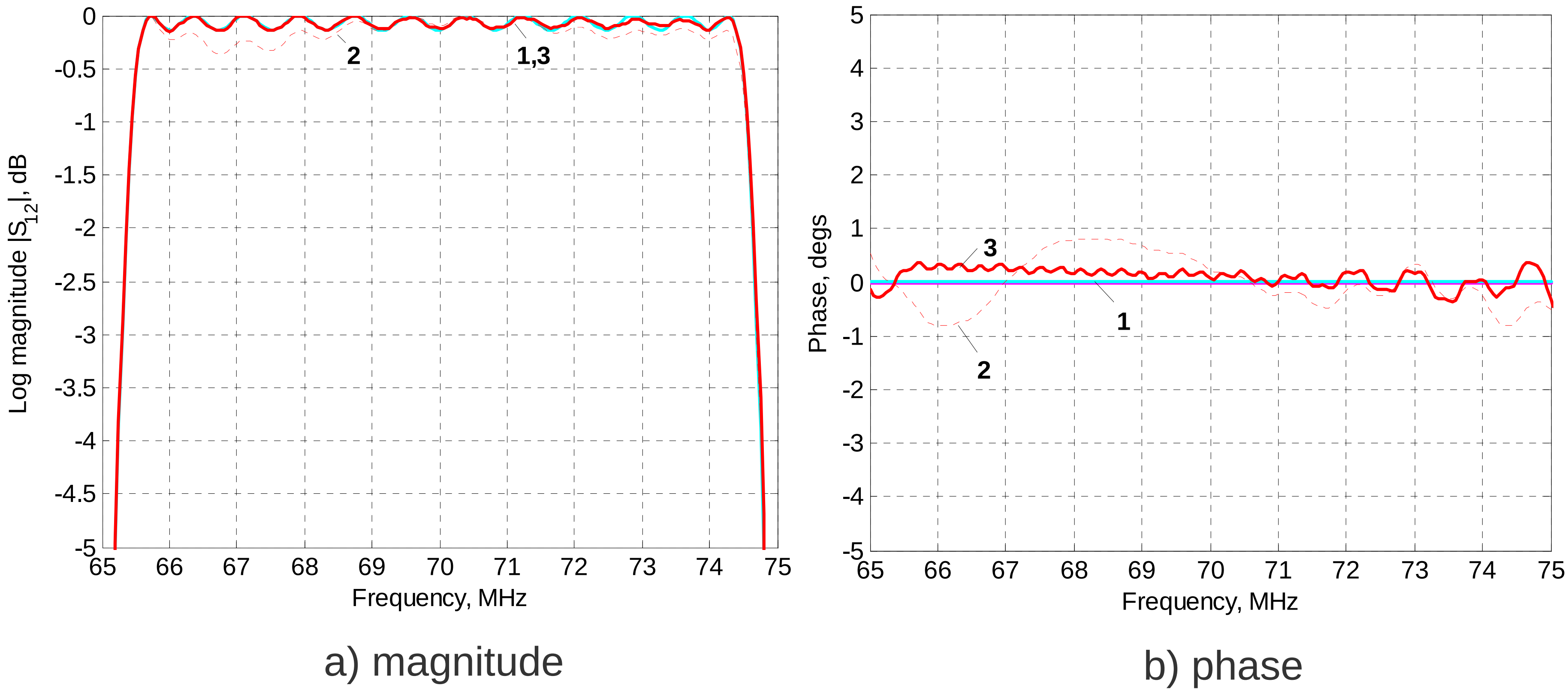


Fig. 15. SAW filter passband response after distortion compensation:
1 – ideal, 2 – unmatched, 3 - matched

Smith Chart

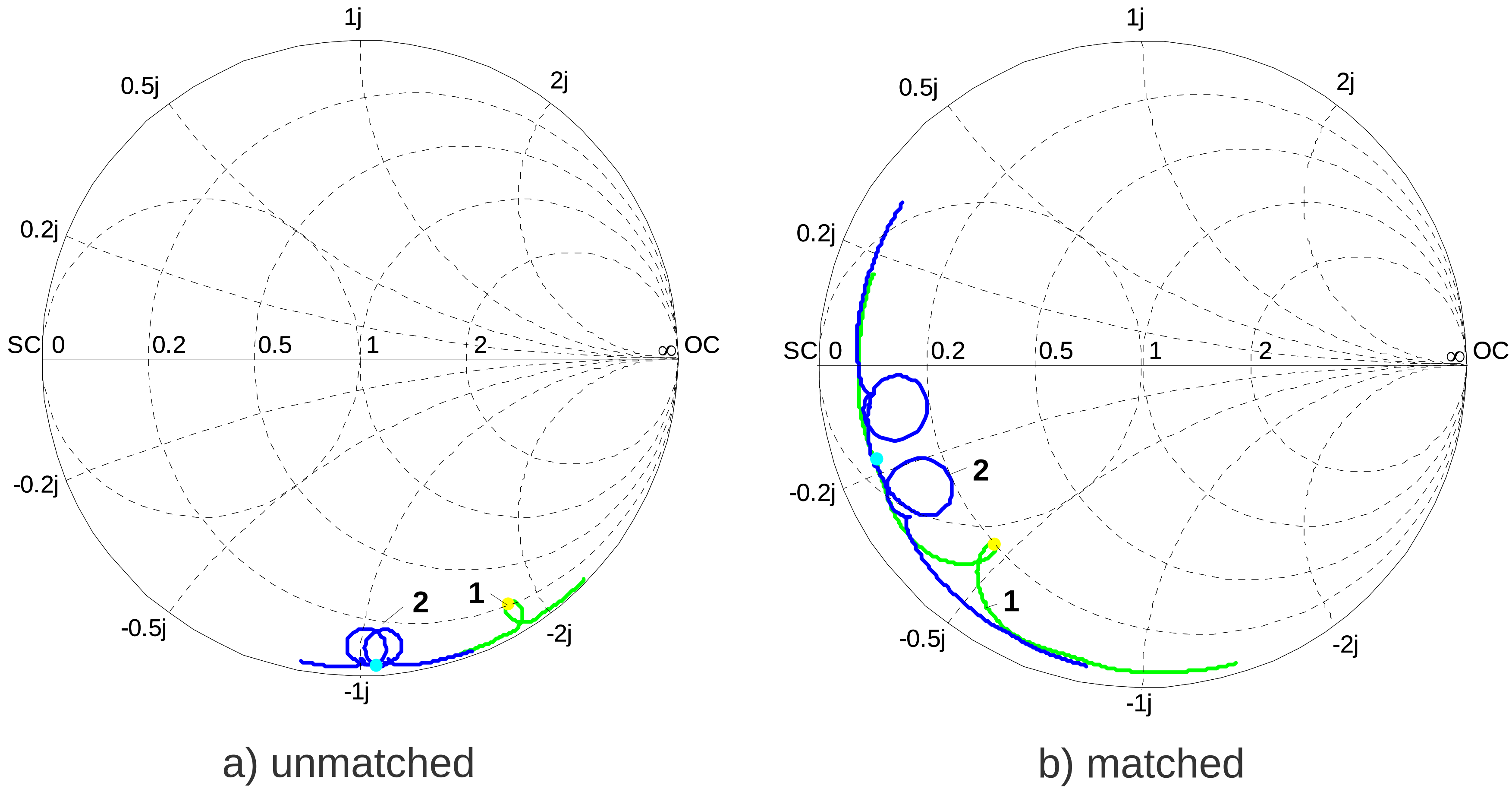


Fig. 16. SAW filter passband response after distortion compensation:
1 – ideal, 2 – unmatched, 3 - matched

Modeled Time Response

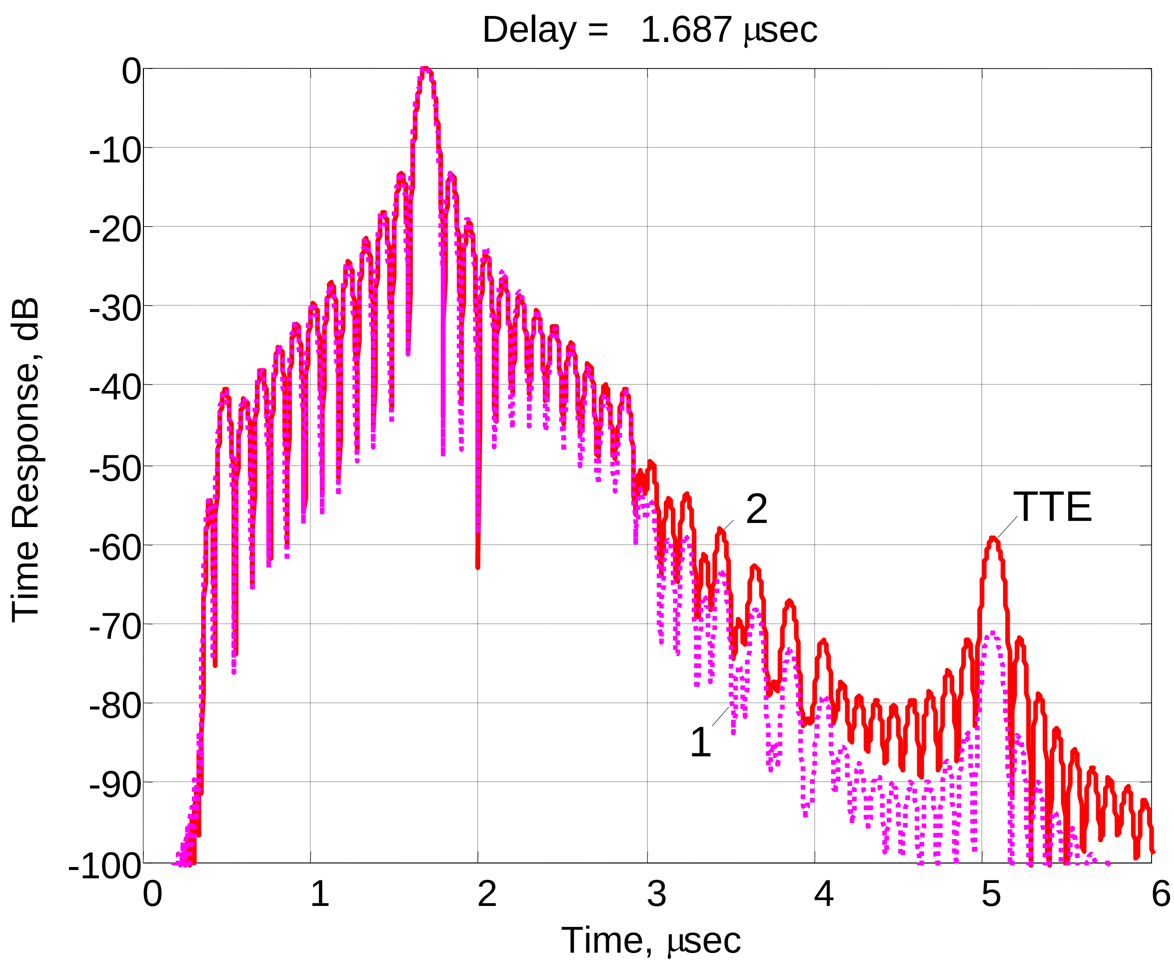


Fig. 17. Modeled time response: 1 – unmatched, 2 – matched

Matching Circuit

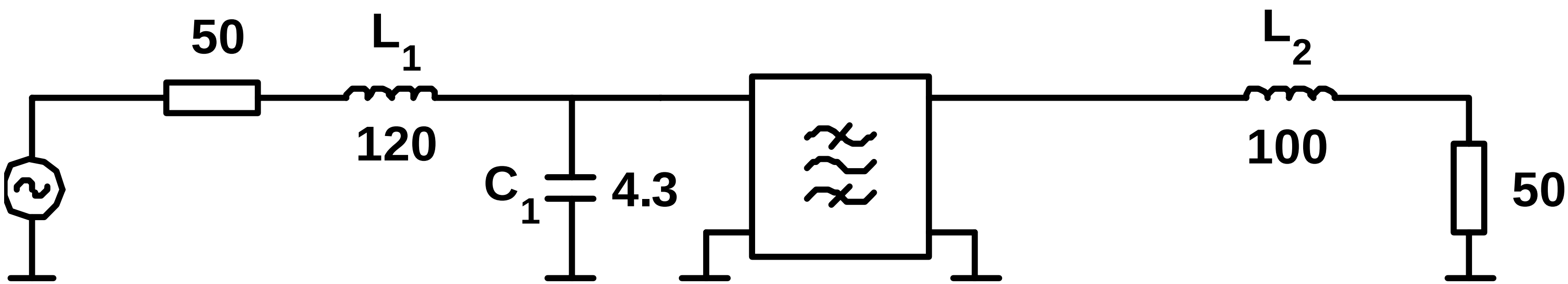


Fig. 18. Matching circuit configuration

Part 5. Modeling of Reflective SAW Transducers

Coupling-of-Modes (COM) Model

Forward and backward traveling surface acoustic waves *a*(x) and *b*(x)

$$a(x) = A(x)e^{-j\frac{K}{2}x}$$
$$b(x) = B(x)e^{+j\frac{K}{2}x} \qquad (83)$$

where *A*(x) and *B*(x) are the slowly varying complex amplitudes and $K=2\pi/p$ is the grating wave number.

For a reciprocal and lossless SAW transducer, the wave excitation and propagation are described by the following system of differential COM-equations

$$\begin{cases} \frac{dA}{dx} = -j\delta A + j\kappa B + j\zeta V \\ \frac{dB}{dx} = -j\kappa^* A + j\delta B - j\zeta^* V \\ \frac{dI}{dx} = -2j\zeta^* A - 2j\zeta B + j\omega C V \end{cases} \qquad (84)$$

COM-Parameters

$\delta = k - K/2$	detuning parameter
k	unperturbated SAW wave number
κ	coupling coefficient
ζ	excitation function
C	static capacitance per unit length
I, V	SAW transducer current and voltage at the electric port

COM Mixed Scattering Matrix

The solution of the system of COM-equations (84) can be found in the closed-form by imposing the boundary conditions on acoustic and electric ports. The elements of the mixed scattering matrix take the form

$$m_{11} = -j\frac{\kappa^* \sin\Phi}{\Delta} \qquad m_{12} = m_{21} = (-1)^N \frac{\gamma}{\Delta}$$

$$m_{13} = \frac{1}{\Delta}\left(\zeta^* \sin\Phi + j\gamma(1-\cos\Phi)\zeta_1\right) \tag{85}$$

$$m_{23} = \frac{(-1)^N}{\Delta}\left(\zeta \sin\Phi + j\gamma(1-\cos\Phi)\zeta_2\right)$$

$$m_{33} = -\frac{4}{\Delta}(1-\cos\Phi)\gamma\zeta_1\zeta_2 - 2\zeta(\zeta\zeta_1 + \zeta^*\zeta_2)(Np - \frac{\sin\Phi}{\Delta})$$

where $\Delta = \gamma\cos\Phi + j\delta\sin\Phi \quad \gamma^2 = \delta^2 - |\kappa|^2 \quad \Phi = N\gamma p \quad \kappa = \dfrac{jr^*}{p}$

$$\zeta = \frac{\xi}{2p} \approx \frac{\sqrt{\omega W K^2 \varepsilon}}{2p} \qquad \zeta_1 = \frac{\delta\zeta^* + \kappa^*\zeta}{\gamma^2} \qquad \zeta_2 = \frac{\delta\zeta + \kappa\zeta^*}{\gamma^2}$$

COM Example: Reflectivity vs Finger Number

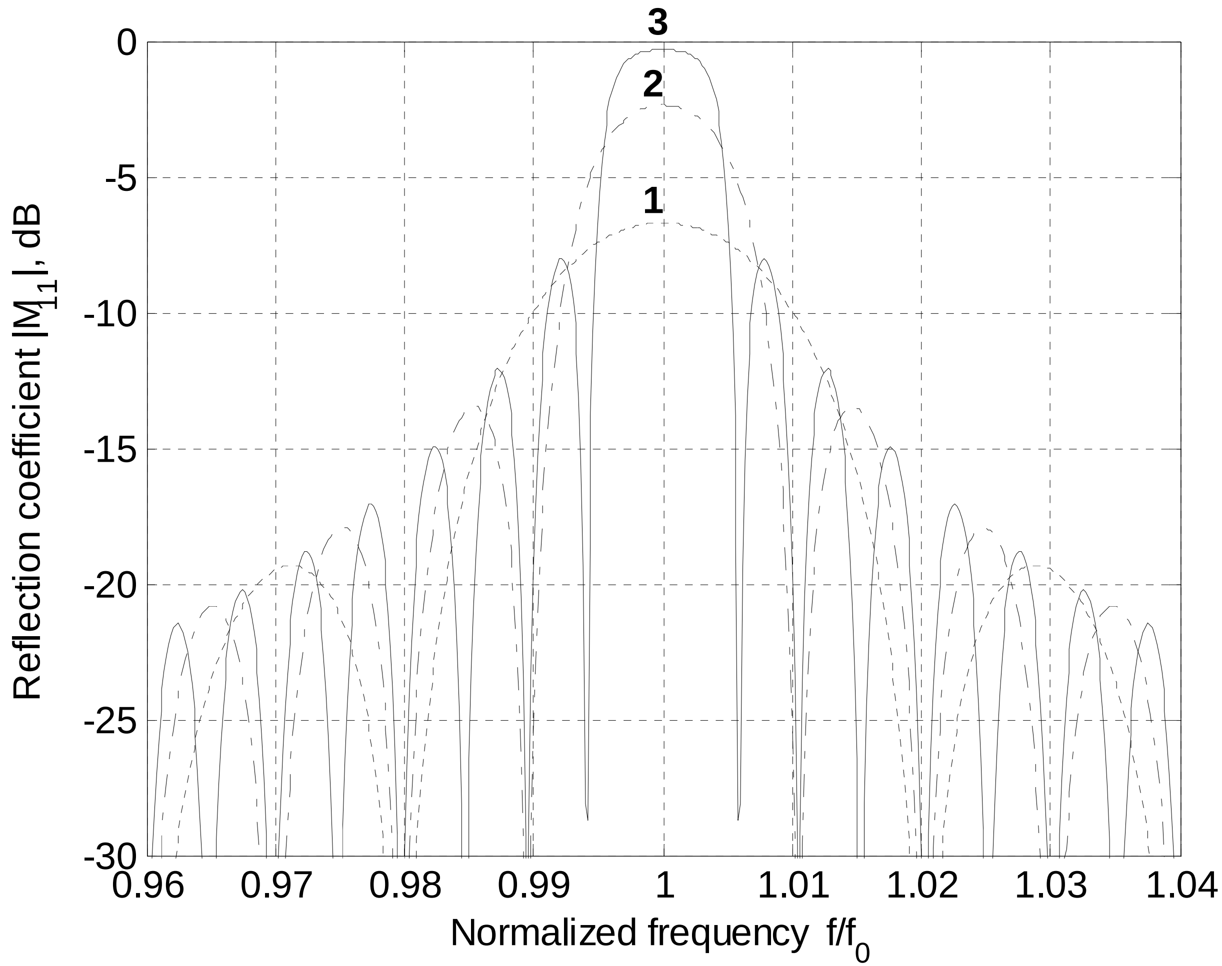


Fig. 19. SAW transducer reflection: finger reflection coefficient *r*=- 0.01*j*
1 – *N*=50, 2 – *N*=100, 3 – *N*=200

COM Example: Reflectivity vs Reflection Coefficient

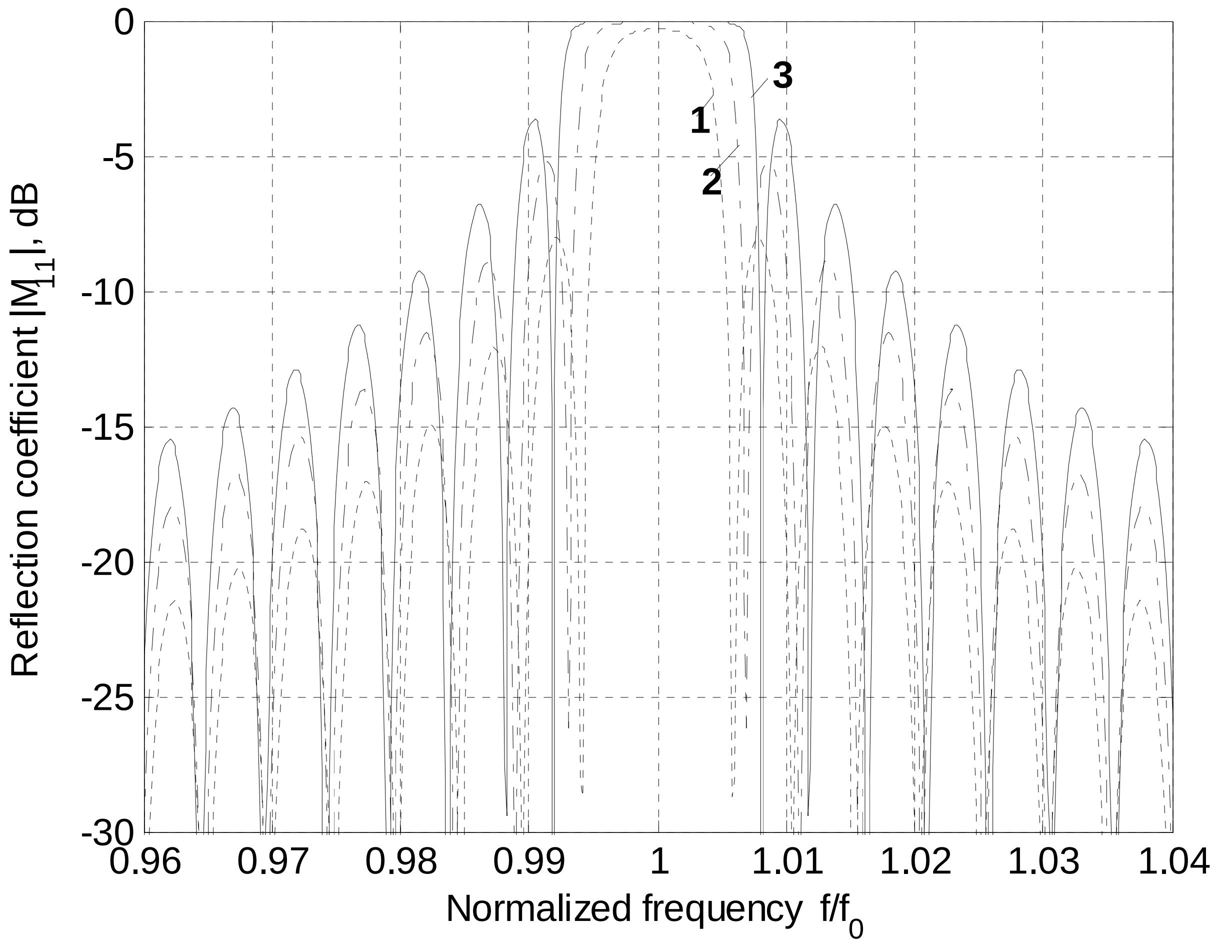


Fig. 20. SAW transducer reflection: number of fingers N= 200, $1 – r$=-0.01j, $1 – r$=-0.015j, $1 – r$=-0.02j

COM Example: Acoustoelectric Conversion

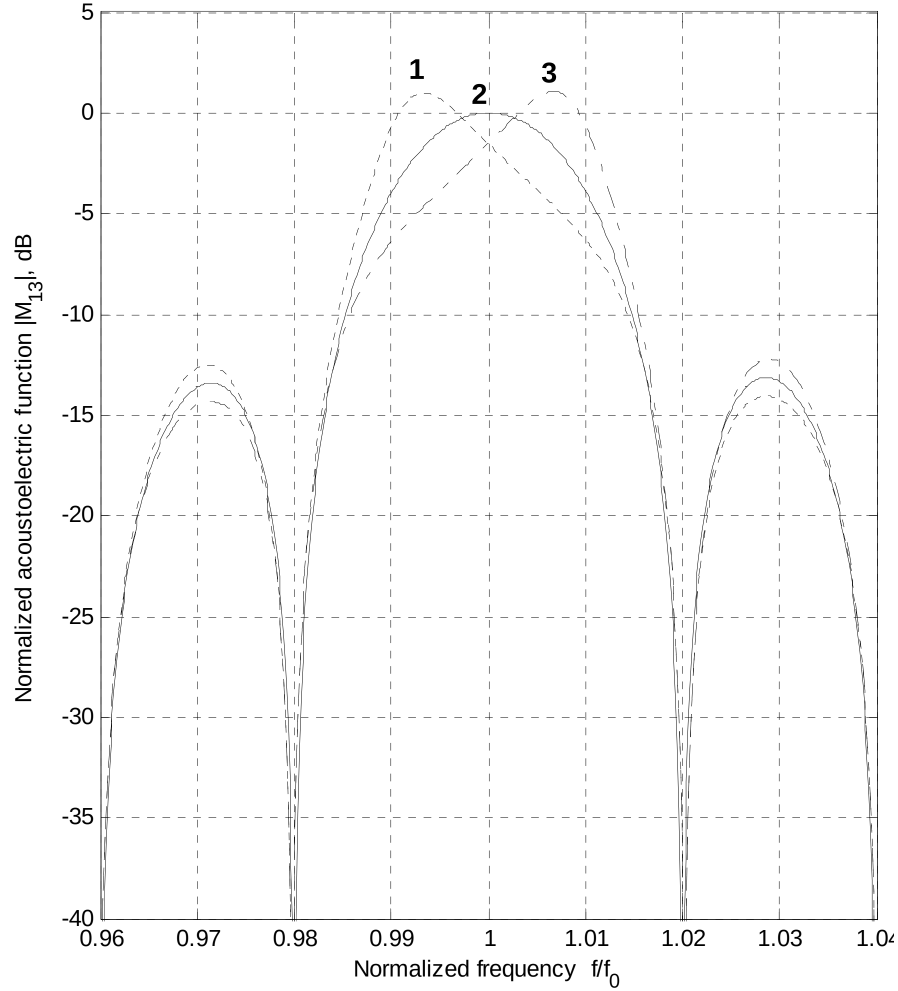


Fig. 21. Acoustoelectric conversion function: number of fingers N=100, 1 – r =-0.01j, 2 – r =0, 3 – r =+0.01j

COM Example: Radiation Admittance

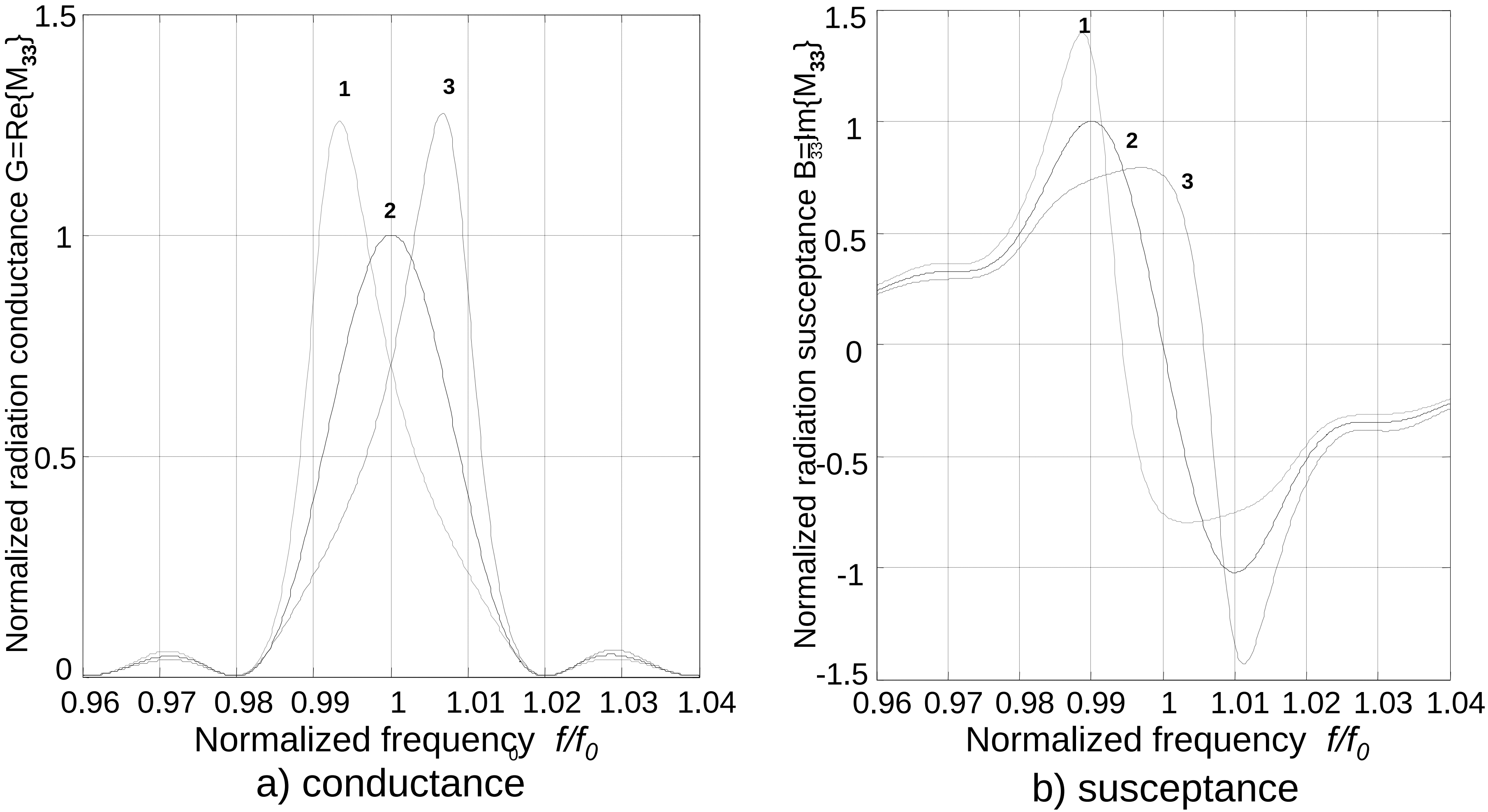


Fig. 21. SAW transducer admittance: number of fingers N=100

1 – r =-0.01j, 2 – r =0, 3 – r =+0.01j

COM Example: Acoustoelectric Conversion

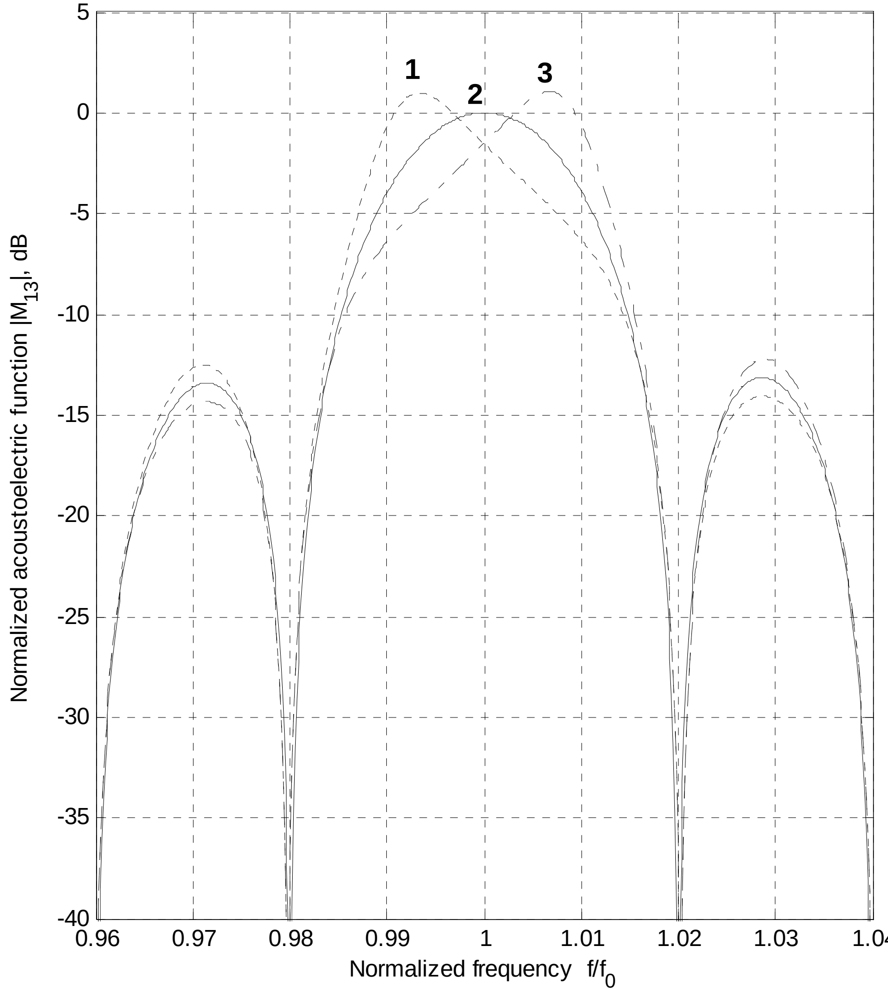


Fig. 21. Acoustoelectric conversion function: number of fingers *N*=100,
1 – *r* =-0.01*j*, 2 – *r* =0, 3 – *r* =+0.01*j*

Conclusions: General

1. The mixed scattering matrix **M** is a convenient and powerful modeling tool for simulation of the acoustoelectric devices comprising acoustic and electric ports.

2. This mixed units matrix combines properties of the classical wave scattering matrix **S** (*S*-parameters) and the admittance matrix **Y** (*Y*-parameters) following by the reciprocity, power conservation, and causality.

3. The mixed scattering matrix **M**, the wave scattering matrix **S**, and the admittance matrix **Y** are interrelated via the closed-form block-matrix equations.

4. A special care must be preserved at selecting and changing the reference planes at the acoustic ports as the change in the reference planes affects the phases of the mixed scattering matrix elements.

5. The closed-form expressions can be simplified by the adequate choice of the reference planes.

Conclusions: SAW Transducer Analysis

1. The mixed scattering matrix of a SAW transducer is a particular case of the mixed scattering matrix, with two acoustic and one electric ports.

2. In general case, the mixed scattering matrix of a SAW transducer contains three independent elements m_{11}, m_{13}, and m_{33} to be determined analytically or numerically while the other elements can be found by reciprocity and power conservation.

3. The electroacoustic conversion function m_{13} plays the key role in the SAW transducer simulation.

4. In the quasi-static approximation, the closed-form equations for the mixed scattering elements can be deduced for reflectionless periodic SAW transducers.

5. For reflective unapodized SAW transducers with the regular polarity sequence, the mixed scattering matrix can be deduced in the closed-form from the COM-theory.

Conclusions: SAW Filter Modeling

1. The mixed scattering matrix can be converted to the mixed transmission matrix relating the acoustic waves at the input and output ports as well as the electric current and voltage at the electric port.

2. The overall mixed scattering matrix of the multiport/multitransducer SAW device can be found by the direct interconnecting of the constituent mixed scattering matrices or by the recurrent cascading the mixed transmission matrices.

3. A SAW filter comprising two in-line SAW transducers is fully characterized by the closed-form two-port admittance matrix **Y** (*Y*-parameters) that can be converted to the wave scattering matrix **S** (*S*-parameters).

4. In general case, the SAW filter transmission coefficient S_{12} is no longer proportional the cross-admittance Y_{12} and therefore it has more complicated behavior than just the product of the acoustoelectric conversion functions.

5. For high-quality performance SAW filters, the distortion of the function S_{12} must be accounted and compensated for at the SAW filter synthesis.

Conclusions: Quasi-Static Approximation

1. In the quasi-static approximation for periodic SAW transducers, the acoustoelectric conversion function m_{13} can be represented as the product of the element factor $\xi(\omega)$ and the array factor $F(\omega)$, provided for a sufficient number of guard fingers at each side to suppress the electrostatic end effects.

2. The element factor is the function of the metallization ratio (duty factor) and it does not depend on the particular SAW filter topology.

3. It is the array factor $F(\omega)$ that specifies the frequency selective properties of SAW transducers, with the shape of the frequency response given by the Fourier Transform of a set of the electrode potentials (finger taps) or gap voltages (gap taps).

3. While both the finger and gap taps models give the same results if correctly applied, the gap model is simpler as it excludes implicitly any uniform potential applied to the transducer.

4. The gap element factor has weaker frequency dependence over the wide frequency range if compared to the finger element factor.

Conclusions: Quasi-Static Approximation (Cont'd)

5. The SAW filter *Y*-parameters take the simplest form in the quasi-static approximation where the self-admittances $Y11$ and $Y22$ are virtually the admittances of the input and output SAW transducers while the cross-admittance $Y12$ is given by the product of the acoustoelectric conversion functions of both transducers.

6. Examples of SAW filter modeling are given.

7. Author's design experience has confirmed good correspondence between the modeled and experimental results.

The End

Thanks for your attention.

Questions?